UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 6 June 2003 1.30 to 4.30

PAPER 57

APPLICATIONS OF DIFFERENTIAL GEOMETRY TO PHYSICS

Attempt **FOUR** *questions.*

There are **seven** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Give a brief sketch of the proof of Stokes' Theorem.

(b) The action functional for eleven-dimensional supergravity theory contains a term of the form

$$S(A) = \int_M \big(\frac{1}{2} F \wedge \star F + F \wedge F \wedge A\big),$$

where $F = dA$ is an exact 4-form and the integral is taken over an eleven-manifold M. Obtain the equation of motion for F by varying $S(A)$ with respect to the 3-form A. Explain why the equations of motion are invariant under

$$A \to A + d\Lambda,$$

where Λ is a 2-form, even though the integrand in the action functional is not.

2 (a) Obtain the Cartan-Maurer formula for the exterior derivatives of the left-invariant one forms λ^a of a matrix Lie group. What form does the formula take in terms of the right-invariant one forms ρ^a? Obtain the Lie brackets of the right and left invariant vector fields $\mathbf{L}_a$ and $\mathbf{R}_a$ respectively.

(b) Show that any semi-simple Lie Group is an Einstein space with respect to the Killing metric. Comment briefly on the signature of the metric, the sign of the curvature and the compactness, or otherwise, of the group.

3 (a) Give the definition of a Poisson manifold and show that every symplectic manifold is a Poisson manifold. State Darboux's theorem for a symplectic manifold.

(b) A Lagrangian submanifold L of a $2n$-dimensional symplectic manifold $\{P, \omega\}$ is an n-dimensional submanifold $L \subset P$ such that everywhere on L

$$\omega(X, Y) = 0,$$

for all vectors X and Y in the tangent space of L.

If, in Darboux coordinates $p_i, q^i,\ i = 1, \ldots, n$, an n-dimensional submanifold is given locally by

$$p_i = p_i(q^j),$$

show that it will be Lagrangian if and only if

$$\frac{\partial p_i}{\partial q^j} = \frac{\partial p_j}{\partial q^i}.$$

Deduce that locally, a Lagrangian submanifold may given in terms of a function of the n coordinates q^i by

$$p_i = \frac{\partial S}{\partial q^i}.$$

4 (a) Define a principal fibre bundle $\{E, \pi, B, G\}$ and show that every such bundle admits a global right action of the structural group G. Illustrate your answer by explaining why $SO(4, 1)$ may be thought of as the bundle of ortho-normal frames for de-Sitter spacetime.

(b) Give an account of connections on a principal bundle in terms of horizontal lifts. Illustrate your answer by means of the example of a rigid convex body rolling without slipping on a plane.

5 (a) Suppose that a Lie group G acts on a symplectic manifold $\{P, \omega\}$ preserving the symplectic form ω. Define the associated moment map $\mu : P \to \star\mathfrak{g}$. Give a condition under which the Poisson algebra coincides with the Lie algebra of G. Show that this condition is satisfied if G is semi-simple.

(b) Given a Hamiltonian system in 2n-variables invariant under the action of $SO(2)$, show how to obtain a reduced Hamiltonian system in $2n - 2$ variables by performing a Marsden-Weinstein reduction. Illustrate your answer using the example of a particle moving in the plane under the influence of a central force.

6 (a) Derive the following isomorphisms.

(i)$SO(3) \equiv SU(2)/\mathbb{Z}_2$

(ii) $SO(2, 1) \equiv SL(2, \mathbb{R})/\mathbb{Z}_2 \equiv Sp(2, \mathbb{R})/\mathbb{Z}_2$

(iii) $SO(3, 1) \equiv SL(2, \mathbb{C})/\mathbb{Z}_2$

(iv) $SO(3, 2) \equiv Sp(4, \mathbb{R})/\mathbb{Z}_2$.

(b) Show that the 3-sphere may be identified with $SU(2)$ and three-dimensional anti-de-Sitter spacetime with $SL(2, \mathbb{R})$. What are the isometry groups of these spaces and how does the isometry group act?

7 Write an essay on Geometric Quantization.