UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 9 June 2003 1.30 to 4.30

PAPER 56

BLACK HOLES

Attempt **THREE** *questions.*

There are **four** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Write an essay describing the proof of the positive energy theorem.

2 The metric on de Sitter space is given by

$$ds^2 = -\left(1 - \frac{1}{3}\Lambda r^2\right) dt^2 + \left(1 - \frac{1}{3}\Lambda r^2\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

where (θ, ϕ) are polar co-ordinates on a sphere, r is a radial coordinate ($0 < r < \infty$), and $\Lambda > 0$ is the cosmological constant. It is known that de Sitter spacetime has a positive cosmological constant and a cosmological horizon at

$$r = \sqrt{3/\Lambda}\,,$$

and that cosmological horizons are like black hole horizons in that they have temperature and entropy.

By making an analytic continuation to Euclidian time τ, find the periodicity of τ required to make the space $r \leqslant \sqrt{3/\Lambda}$ free of singularities.

What is the temperature of the cosmological horizon?

The action

$$I = \frac{-1}{16\pi} \int (R - 2\Lambda) g^{1/2} d^4x$$

when varied with respect to the metric produces a field equation satisfied by de Sitter space.

Find this field equation.

Hence (assuming there are no boundary correction terms) find the Euclidean action for de Sitter space.

de Sitter space is a vacuum spacetime that contains no localized mass points, and so its internal energy can be taken to be zero.

Use your previous results to calculate the entropy of the cosmological horizon.

3 The metric of the Schwarzschild spacetime is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,.$$

Find the advanced and retarded Eddington-Finkelstein co-ordinates, and use them to show that $r = 2m$ is only a co-ordinate singularity.

Describe how these results lead to the concept of white hole and black hole horizons.

Construct Kruskal co-ordinates for this spacetime, and find the metric in these co-ordinates.

Use your results to construct the Penrose diagram for the Schwarzschild spacetime, and comment about the various infinities, horizons and singularities in your diagram.

4 The Reissner-Nordstrom metric is given by

$$ds^2 = -\Big(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\Big)dt^2 + \Big(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\Big)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,.$$

What are the conditions on M and Q that are necessary for this spacetime to be a black hole?

Show that

$$k^a = (1, 0, 0, 0)$$

satisfies Killing's equation.

Why would one expect null lines that lie on the black hole horizon to be both geodesic and have tangent vectors that are Killing vectors.

Explain why one then expects that

$$k^a \nabla_a k_b \quad = \quad \pm\kappa k_b$$

where κ is a positive constant.

Evaluate κ and comment on the possibility that it might vanish.