UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 5 June 2003 1.30 to 3.30

PAPER 55

COSMOLOGY

Attempt **THREE** *questions.*

There are **four** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 In the early Universe the Friedmann equation and acceleration equation are

$$H^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2},$$
$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)\,.$$

The Universe contains a perfect fluid with equation of state $P = (\gamma - 1)\rho$. Show that

$$\mathcal{H}' + \alpha(\mathcal{H}^2 + k) = 0,$$

where $\alpha = (3\gamma - 2)/2, \mathcal{H} = a'/a$ and prime denotes derivative with respect to conformal time τ. Show

$$\mathcal{H} = \begin{cases} \cot(\alpha\tau) & k = +1 \\ 1/(\alpha\tau) & k = 0 \\ \coth(\alpha\tau) & k = -1 \end{cases}$$

Derive expressions for $a(\tau)$ and $t(\tau)$. Hence compare the age of the Universe in the matter era for an open, closed and flat Universe.

2 Briefly discuss the evidence for a net baryon asymmetry in the Universe. Discuss the necessary conditions to create a baryon asymmetry and show how baryogenesis can be implemented in a simple example.

3 In the early Universe, the Hubble constant is

$$H = 1.66 g^{*1/2} T^2 / m_{pl}\,.$$

Consider the synthesis of light elements. For temperatures $T > 1MeV$ neutrons and protons are in thermal equilibrium via interactions $\nu + n \leftrightarrow p + e^-$, with the interaction rate $\Gamma = G_F^2 T^5$, where the Fermi constant is $G_F \approx 10^{-5} m_p^{-2}$ and $m_p \approx 1GeV$. Estimate the decoupling temperature, T_D, of neutrinos given that the effective degree of freedom at this temperature is $g^* = 10.75$. Show that the neutron to proton density is given by

$$\frac{n_n}{n_p} = \frac{X_n}{X_p} = \exp(-Q/T),$$

where $X_i = n_i/n_B$, with n_B being the baryon number of the Universe and $Q = m_n - m_p$.

Subsequently deuterium is produced via the interaction $p + n \leftrightarrow D + \gamma$. Estimate the fractional deuterium abundance, X_D, in terms of the binding energy, $B = m_D - m_n - m_p$ and the ratio $\eta = n_B/n_\gamma$.

Explain why the resulting ^4He abundance is sensitive to the neutron half-life and the effective number of degrees of freedom. In some extensions to general relativity Newton's constant varies. How would this affect the helium abundance?

4 In the early Universe the equations of motion of a scalar field are

$$H^2 = \frac{8\pi G}{3}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right],$$
$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0,$$

where $V(\phi)$ is the potential. Briefly discuss why it is thought necessary to have an inflationary epoch in the early Universe.

Use the potential

$$V = V_0 \exp\left(-\frac{c\phi}{m_{pl}}\right)$$

as an example. Find the solution for the field and the scale factor and show how cosmological problems are solved. What are the problems with this model?

[*Hint: Try a solution of the form* $\phi = \alpha\ ln\ (\beta t)$.]