UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 3 June 2003 1.30 to 4.30

PAPER 54

GENERAL RELATIVITY

Attempt **THREE** *questions.*

There are **four** *questions in total.*

The questions carry equal weight.

Information

The signature is $(+\ -\ -\ -)$*, all connections are symmetric, and the curvature tensor conventions are defined by*

$$R^i{}_{kmn} = \Gamma^i{}_{km,n} - \Gamma^i{}_{kn,m} - \Gamma^i{}_{pm}\Gamma^p{}_{kn} + \Gamma^i{}_{pn}\Gamma^p{}_{km}\,.$$

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 The *Lie derivative with respect to a vector field* X of a scalar function f and of a vector field Y are defined to be

$$\mathcal{L}_X f = X(f), \qquad \mathcal{L}_X Y = [X, Y],$$

where $[X, Y]$ is the *commutator of* X *and* Y. Write out these definitions in suffix notation and extend the definition of the Lie derivative to cover covector and $\binom{1}{1}$ tensor fields. Explain briefly how you would extend the definition to $\binom{m}{n}$ tensor fields.

Show that

$$\mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X = \mathcal{L}_{[X,Y]}.$$

Explain what is meant by a *Killing vector*. Show that a linear combination of Killing vectors with constant coefficients is a Killing vector. Show also that the commutator of two Killing vectors is a Killing vector.

The metric for an axially symmetric rotating star admits precisely two linearly independent Killing vectors T and Φ. Far from the star where fields are weak, and using a standard cylindrical polar chart (t, r, z, φ) we have

$$T \to \frac{\partial}{\partial t}, \qquad \Phi \to \frac{\partial}{\partial \varphi}, \qquad \text{as } (r^2 + z^2) \to \infty.$$

Show that T and Φ commute.

[*You may quote the Jacobi identity.*]

2 Write an essay on gauge-invariant linearized vacuum perturbations of flat Minkowski spacetimes.

[*You may use any information from the lecture handout included with this examination paper.*]

3 Using standard notation the action S for sourcefree electromagnetism is given by

$$S = -\frac{1}{16\pi} \int \sqrt{-g}\, F^{ab} F_{ab}\, d\Omega,$$

where $F_{ab} = A_{b;a} - A_{a;b}$ and A_a is the vector potential. Show first that $F_{[ab;c]} = 0$.

Next derive the field equation

$$F^{ab}{}_{;b} = 0,$$

by variation of S with respect to A_a. Obtain the energy-momentum tensor T^{ab} from a variational principle and show that requiring $T^{ab}{}_{;b} = 0$ implies the same field equation provided that $F^a{}_c$, regarded as a matrix, is non-singular.

Show that an equivalent action is

$$S = \frac{1}{16\pi} \int \sqrt{-g} \left[F^{ab} F_{ab} - 2F^{ab}(A_{b;a} - A_{a;b}) \right] d\Omega,$$

where F^{ab} is antisymmetric and both F^{ab} and A_a must be varied independently, the *Palatini procedure.*

[*You may assume the principle of equivalence.*]

4 Let $d\Sigma^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ denote the line element on the unit 2-sphere, and let

$$\widehat{ds}^{\,2} = dt^2 - dr^2 - r^2\, d\Sigma^2$$

denote the Minkowski spacetime line element in spherical polar coordinates. Introduce retarded coordinates $u = t - r$ and $v = t + r$ and make a conformal transformation to an unphysical spacetime with line element ds^2 given by $ds^2 = 4(1+u^2)^{-1}(1+v^2)^{-1}\widehat{ds}^{\,2}$. Perform further coordinate changes $p = \tan^{-1} u$, $q = \tan^{-1} v$, followed by $T = q + p$, $R = q - p$ to obtain

$$ds^2 = dT^2 - dR^2 - \sin^2 R\, d\Sigma^2,$$

where the ranges of all of the coordinates t, r, θ, ϕ, u, v, p, q, T and R should be stated explicitly.

Use your results to discuss the asymptotic behaviour of Minkowski spacetime geodesics as seen in the unphysical spacetime.