UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 10 June 2003 1.30 to 3.30

PAPER 52

SOLITONS AND INSTANTONS

Attempt **THREE** *questions.*

There are **four** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A static, real scalar field $\phi(x)$ defined on $\mathbb{R}$ has potential energy function

$$E = \int_{-\infty}^{\infty} \left(\frac{1}{2} \left(\frac{d\phi}{dx} \right)^2 + U(\phi) \right) \, dx \, .$$

Derive the field equation satisfied by ϕ.

Suppose the field equation is

$$\frac{d^2\phi}{dx^2} = \phi - 4\phi^3 + 3\phi^5 \, .$$

Find the potential $U(\phi)$ in this case, choosing the arbitrary constant so that the minimum of U is zero. What are the vacua of the theory? How many types of kink solution are expected to exist in this theory? Do they all have the same mass?

Find the Bogomolny equation satisfied by one of the kink solutions. Using a Bogomolny argument, or otherwise, find the mass of this kink. Find the explicit kink solution.

2 The static, critically coupled abelian Higgs model in the plane has a complex scalar field ϕ and a $U(1)$ gauge potential a_j $(j = 1, 2)$. The potential energy function is

$$E = \int_{\mathbb{R}^2} \left(\frac{1}{2} B^2 + \frac{1}{2} (D_j \phi)^* D_j \phi + \frac{1}{8} (1 - \phi^* \phi)^2 \right) \, d^2 x$$

where $B = \partial_1 a_2 - \partial_2 a_1$, and $D_j \phi = \partial_j \phi - i a_j \phi$. Explain how finite energy field configurations are characterized by an integer topological charge N. Discuss properties of the field configuration that depend on N.

There is a vortex solution which in polar coordinates (r, θ) has the form

$$\begin{aligned} \phi(r, \theta) &= h(r) e^{i\theta} \\ a_r(r, \theta) &= 0 \\ a_\theta(r, \theta) &= a(r) \end{aligned}$$

for certain real functions $h(r)$ and $a(r)$. The energy in terms of $h(r)$ and $a(r)$ is

$$E = \pi \int_0^\infty \left(\frac{1}{r^2} \left(\frac{da}{dr} \right)^2 + \left(\frac{dh}{dr} \right)^2 + \frac{1}{r^2} (1 - a)^2 h^2 + \frac{1}{4} (1 - h^2)^2 \right) r \, dr \, .$$

An approximation to the vortex is obtained by setting

$$h(r) = \begin{cases} \frac{r}{r_0} & 0 \leqslant r \leqslant r_0 \\ 1 & r \geqslant r_0 \end{cases}$$

$$a(r) = \begin{cases} \left(\frac{r}{r_0} \right)^2 & 0 \leqslant r \leqslant r_0 \\ 1 & r \geqslant r_0 \, . \end{cases}$$

What is the topological charge of the vortex? Calculate the energy of this approximate vortex field, and by optimizing the choice of r_0, estimate the energy of the vortex.

3 Describe how the Derrick scaling argument is used to determine whether or not topological soliton solutions may exist in a field theory. If solutions do exist, what can be learnt about them using the scaling argument?

An $SU(2)$ sigma model, defined in $(d+1)$-dimensional Minkowski space, has a scalar field $U(x^0, \mathbf{x})$ taking values in the Lie group $SU(2)$. The field satisfies the boundary condition $U \to 1_2$ as $|\mathbf{x}| \to \infty$, where 1_2 is the identity of $SU(2)$. The current R_μ is defined by $R_\mu = (\partial_\mu U)U^{-1}$, and takes values in the Lie algebra of $SU(2)$. The Lagrangian density is

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}(R_\mu R^\mu) \, .$$

What can you learn about soliton solutions in this model using topological ideas, and the Derrick scaling argument?

How do your conclusions change if the sigma model is replaced by the Skyrme model, whose Lagrangian density is

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}(R_\mu R^\mu) + \frac{1}{16}\mathrm{Tr}([R_\mu, R_\nu][R^\mu, R^\nu]) \, ?$$

[The indices μ, ν run through $0, 1, \ldots, d$. Indices are raised with the Minkowski metric $\eta^{\mu\nu} =$ diag $(1, -1, \ldots, -1)$. $[\, , \,]$ denotes the bracket of the $SU(2)$ Lie algebra and Tr denotes the trace.]

4 Write an essay on "instantons".