UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 4 June 2003 9 to 12

PAPER 51

CONFORMAL FIELD THEORY

Attempt **FOUR** *questions.*

There are **six** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Establish the condition for the infinitesimal transformation $x^\mu \to x^\mu + \epsilon\omega^\mu(x) + \mathcal{O}(\epsilon^2)$ of flat d-dimensional Euclidean space to be conformal. If this condition is satisfied and $\rho = \partial_\alpha \omega^\alpha$, show that

$$\frac{\partial^2 \omega^\alpha}{\partial x^\beta \partial x^\gamma} = \frac{1}{d}\left(\frac{\partial \rho}{\partial x^\gamma}\,\delta_{\alpha\beta} + \frac{\partial \rho}{\partial x^\beta}\,\delta_{\alpha\gamma} - \frac{\partial \rho}{\partial x^\alpha}\,\delta_{\beta\gamma} \right).$$

Show that $\partial^2\rho = 0$ if $d > 1$, and $\partial_\alpha\partial_\beta\rho = 0$ if $d > 2$. Deduce the general form of ω for $d > 2$. What is the dimension of the group of conformal transformations in this case?

2 Give an account of the relationship of the energy-momentum tensor of a field theory to conformal transformations of the theory, with particular reference to two-dimensional space-time, illustrating your remarks by reference to the theory of a free scalar field.

3 Describing two-dimensional Euclidean space by the complex variable $z = x + iy$, suppose that the Euclidean quantum fields $\phi_j(z)$ satisfies

$$U_\gamma \phi_j(z) U_\gamma^{-1} = \left(\frac{d\zeta}{dz}\right)^{h_j} \phi_j(\zeta),$$

where U_γ is an operator defined for each Möbius transformation $\zeta = \gamma(z) = (az+b)/(cz+d)$. Suppose further that U_γ is a symmetry of the vacuum for each γ, *i.e.* $U_\gamma|0\rangle = |0\rangle$ and $\langle 0|U_\gamma = \langle 0|$. Show that, for $n = 1, 2$ and 3, the vacuum expectation values,

$$\langle 0|\phi_1(z_1)\phi_2(z_2)\ldots\phi_n(z_n)|0\rangle,$$

are determined up to a multiplicative constant. What are the contraints on h_1, h_2 and h_3 for these vacuum expectation values to be nonzero? What can be said for $n > 3$?

4 Show that the free scalar field theory defined by the action

$$S[\phi] = \int \tfrac{1}{2}\partial^\mu\phi\,\partial_\mu\phi\,d^dx$$

is conformally invariant in d-dimensional space-time. What is special about $d = 2$? In this case, assuming the periodicity condition $\phi(x+\ell, t) = \phi(x, t)$, describe the quantization of the theory and show how it leads to a representation of the Virasoro algebra.

5 Describe the irreducible unitary highest weight representations of the Virasoro algebra with central charge $c < 1$. Show explicitly how to construct these representations for $c = \frac{1}{2}$.

6 Starting from a representation of the affine algebra,

$$[T^a_m, T^b_n] = if^{abc}T^c_{m+n} + km\delta_{m,-n}\delta^{ab},$$

where f^{abc} is totally antisymmetric and $[t^a, t^b] = if^{abc}t^c$ is a simple Lie algebra, g, show how to contruct a representation of the Virasoro algebra, with the value of the central charge, c, of the Virasoro algebra given by

$$c = c^g \equiv \frac{2k \dim g}{2k + Q^g},$$

where $f^{abc}f^{abd} = Q^g\delta^{cd}$, so that Q^g is the value of the quadratic Casimir operator in the adjoint representation of g. Explain what is meant by a Cartan subalgebra h of g. By considering also the coset construction associated to the pair $g \supset h$, or otherwise, show that

$$\dim g \geqslant c^g \geqslant \text{rank } g.$$