UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 6 June 2003 9 to 12

PAPER 50

STRING THEORY

Attempt **THREE** *questions.*

There are **five** *questions in total.*

The questions carry equal weight.

Candidates should note that trivial errors in numerical factors will not be penalized.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Write down the action that describes the dynamics of a relativistic bosonic string moving in flat d-dimensional Minkowski space-time. How is the string coupling constant related to the topology of the world-sheet?

What are the local symmetries of this action? Outline how these can be used to fix the conformal gauge, leading to the presence of Faddeev–Popov ghost and anti-ghost fields in the quantum gauge-fixed action (you need not give a detailed discussion of the Faddeev–Popov formalism).

The BRST charge, Q, generates a fermionic symmetry of the gauge-fixed action and is nilpotent ($Q^2 = 0$) when $d = 26$. How may Q be used to define physical states of the string?

The contribution of the modes of the ghost and anti-ghost fields (c_m and b_m) to the Virasoro generators is given by $L_m^{(b,c)} = \sum_n (m-n)\, b_{m+n} c_{-n}$. Show that

$$[L_m^{(b,c)}, b_n] = (m-n)\, b_{m+n} \qquad \text{and} \qquad [L_m^{(b,c)}, c_n] = -(2m+n)\, c_{m+n}.$$

For the open bosonic string Q has the form

$$Q = \sum_n : c_n \left(L_{-n}^{\alpha} + \tfrac{1}{2} L_{-n}^{(b,c)} \right) : -c_0 \,,$$

where L_m^{α} are the Virasoro generators obtained from the world-sheet bosons. Show that physical states $|\phi\rangle$, satisfying

$$b_n|\phi\rangle = 0 = c_n|\phi\rangle \,, \quad \text{for } n > 0 \,, \qquad\qquad b_0|\phi\rangle = 0 \,,$$

are those that satisfy the standard Virasoro conditions.

[*You may use the anticommutation relations* $\{b_m, c_n\} = \delta_{m+n,0}$.]

2 The action for a bosonic string propagating in 26-dimensional Minkowski space-time is given in the conformal gauge by

$$A[X] = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma\, \partial_\alpha X^\mu \partial^\alpha X_\mu$$

where $\alpha = 0, 1$ and $\mu = 0, 1, \ldots, 25$. Determine the general solution of equations of motion for the closed string, thereby deducing the expansion of X^μ in terms of the left-moving and right-moving mode coeffients, α^μ_m and $\tilde{\alpha}^\mu_m$.

Write down the closed-string physical state conditions in terms of Virasoro operators, $L_m = \frac{1}{2}\sum_n : \alpha_{m-n} \cdot \alpha_n :$. Show that the massless states of the closed bosonic string are given by $\zeta_{\mu\nu}\, \alpha^\mu_{-1}\tilde{\alpha}^\nu_{-1}\, |0; p\rangle$, where $|0; p\rangle$ is the ground state of momentum p^μ satisfying $\alpha' p^2 = 4$ and $\zeta_{\mu\nu}$ satisfies conditions that should be specified.
[*You may use* $[\alpha^\mu_m, \alpha^\nu_n] = m\, \eta_{\mu\nu}\, \delta_{m+n,0} = [\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n]$ *and* $[\alpha^\mu_m, \tilde{\alpha}^\nu_n] = 0$, *where* $\eta_{\mu\nu}$ *is the Minkowski space-time metric.*]

Now consider the situation in which the X^1 is periodic so that $X^1 \equiv X^1 + 2\pi R$ where R is the radius of the periodic dimension. How does the mode expansion for X^1 depend on R?

Enumerate the massless states for generic values of R. Find the extra massless states that arise at the 'self-dual' point, $R = \alpha'/R$.

3 The amplitude for the scattering of M ground states (tachyons) of the closed bosonic string is given by the functional integral

$$T_M(\{k_r\}) = \int \prod_{r=1}^{M} d^2 z_r \int DX \prod_{r=1}^{M} V_0(k_r, z_r) \exp(-A[X]) \rangle \,.$$

In this expression $A[X]$ is the bosonic string action on a flat euclidean world-sheet that spans the complex plane and each tachyon vertex operator is given by $V_0(k_r, z_r) = e^{ik_r \cdot X(z_r)}$ where z_r is the complex position of the operator on the world-sheet and k_r^μ is the momentum of the r'th tachyon state (and momentum conservation implies that $\sum_r k_r^\mu = 0$).

Explain how the amplitude can be written in the form

$$T_M(\{k_r\}) = \mathcal{N} \int \prod_{r=1}^{M} d^2 z_r \prod_{r<s} |z_r - z_s|^{\alpha' k_r \cdot k_s} \,,$$

where $\mathcal{N}$ is an irrelevant normalization constant.

Show that, provided $\alpha' k_r^2 = 4$, the integrand of the expression for A_M is invariant under $SL(2, C)$ transformations

$$z_r \to \frac{a z_r + b}{c z_r + d} \,,$$

where a, b, c, d are complex and satisfy $ad - bc = 1$. Indicate how this symmetry can be used to fix $z_1 = -\infty$, $z_2 = 0$, $z_4 = 1$, and hence show that the four-tachyon amplitude reduces to (up to a momentum-independent factor)

$$T_4 = \frac{\Gamma(-1 - \frac{\alpha'}{4}s)\Gamma(-1 - \frac{\alpha'}{4}t)\Gamma(-1 - \frac{\alpha'}{4}u)}{\Gamma(2 + \frac{\alpha'}{4}s)\Gamma(2 + \frac{\alpha'}{4}t)\Gamma(2 + \frac{\alpha'}{4}u)} \,,$$

where the Mandelstam invariants $s = -(k_1 + k_2)^2$, $t = -(k_1 + k_4)^2$ and $u = -(k_1 + k_3)^2$ satisfy $s + t + u = -16/\alpha'$. Comment on the singularities of this expression as a function of s, t and u.

[*You may use*

$$\int d^2 z \, |z|^{-A} \, |1 - z|^{-B} = \pi \frac{\Gamma(1 - \frac{1}{2}A)\Gamma(1 - \frac{1}{2}B)\Gamma(1 - \frac{1}{2}C)}{\Gamma(2 - \frac{1}{2}(A + B))\Gamma(2 - \frac{1}{2}(B + C))\Gamma(2 - \frac{1}{2}(C + A))} \,,$$

where $C = 4 - A - B$.]

4 Consider the theory obtained by adding the free Dirac action for world-sheet Majorana fermion fields, $\psi^\mu_a(\sigma,\tau)$ ($a = 1, 2$ being the world-sheet spinor index), to the bosonic string action in conformal gauge,

$$A = -\frac{1}{2\pi}\int d\tau d\sigma\ \left(\partial_+ X_\mu \partial_- X^\mu - \psi^\mu_2 \partial_+ \psi_{2\,\mu} - \psi^\mu_1 \partial_- \psi_{1\,\mu}\right)\,,$$

where $\partial_\pm = \partial_\tau \pm \partial_\sigma$. Show that the combined action possesses a global fermionic symmetry (world-sheet supersymmetry) under the transformations

$$\delta X^\mu = \epsilon_1 \psi^\mu_1 + \epsilon_2 \psi^\mu_2, \qquad \delta\psi^\mu_1 = \partial_+ X^\mu \epsilon_1 \qquad \delta\psi^\mu_2 = \partial_- X^\mu \epsilon_2\,,$$

where ϵ_1 and ϵ_2 are Grassmann-valued components of a constant world-sheet spinor parameter.

By varying the fields in the action, show that there are two possible boundary conditions that the fermionic fields for a closed string may satisfy and obtain the equations of motion? Deduce the expansions of these fields in terms of mode coefficients $\psi^\mu_{1\,m}$ and $\psi^\mu_{2\,m}$ and state, without proof, their anti-commutation relations.

In the light-cone gauge in the critical dimension ($d = 10$) the excited physical string states are expressed in terms of the transverse components of the non-zero modes of $\psi^i_a(\sigma,\tau)$ and $X^i(\sigma,\tau)$ ($i = 1,\ldots,8$). For the open string the masses of these states are given by

$$\alpha'(\text{Mass})^2 = N^\alpha + N^\psi - a\,,$$

where $N^\alpha = \sum_{n>0} \alpha^{\dagger\,i}_n \alpha^i_n$, $N^\psi = \sum_{n>0} n\psi^{\dagger\,i}_n \psi^i_n$ and the normal ordering constant takes the values $a = 1/2$ or $a = 0$, depending on the boundary conditions. [Note that for an open string $\psi^\mu_{1\,m} = \psi^\mu_{2\,m} \equiv \psi^\mu_m$.]

Write down all the open-string physical states up to and including the states with $\alpha'(\text{Mass})^2 = 1$. Outline *briefly* how a truncation of the space of states might lead to a theory with no tachyon and with equal numbers of fermionic and bosonic states at each mass level.

5 Answer **one** of the following:

(a) Discuss the behaviour of the density of states of the closed bosonic string at asymptotically large mass levels and the occurence of a phase transition at the Hagedorn temperature.

(b) Explain how general relativity coupled to matter arises from string theory in the 'low energy' limit.