UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 6 June 2003 9 to 12

PAPER 5

GEOMETRIC GROUP THEORY

Attempt **THREE** *questions.*

There are **five** *questions in total.*

The questions carry equal weight.

Notation and Conventions *Let A be a finite set, A^* is the set of all finite words over A. The length of a freely reduced word $w \in A^*$ is denoted $|w|$.*

Let G be a finitely presented group and $\langle A \,|\, R\rangle$ be a finite presentation of G. The Cayley graph associated with the given presentation is denoted by $C(G, A)$.

Given functions $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ we write $f \lesssim g$ if and only if there exists a constant K such that

$$f(n) \leqslant Kg(Kn + K) + (Kn + K).$$

We say that f and g are equivalent written $f \sim_e g$ if and only if

$$g \lesssim f \lesssim g.$$

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let X be a space with $G = \pi_1(X, x_0)$, let H be a subgroup of G.

(i) If $((\widetilde{X}, \widetilde{x}_0), p)$ is a covering space of X prove that $p_* : \pi_1(\widetilde{X}) \to \pi_1(X)$ is a monomorphism. State clearly any results that you use.

(ii) Prove that there is a covering space $(\widetilde{X}(H), p)$ such that $\pi_1(\widetilde{X})$ is isomorphic to H. State clearly any results that you use.

(iii) Prove the Nielsen-Schreier Theorem which states that if F is a free group and H is a subgroup of F then H is a free group.

2 (i) Define a (λ, ϵ) quasi-isometry between metric spaces (X, d_X) and (Y, d_Y), and prove that for any quasi-isometry $f : X \to Y$ there exists a quasi-isometry $f' : Y \to X$.

(ii) Given a finite presentation $\langle A \mid R \rangle$ of a group G define the Cayley graph $C(G, A)$ and the word metric. What does it mean to say that two finitely presented groups are quasi-isometric ? Let $\langle A|R\rangle$ and $\langle B|S\rangle$ be two finite presentations of a group G. Show that the the Cayley graphs $C(G, A)$ and $C(G, B)$ are quasi-isometric.

(iii) Prove that if H is a subgroup of finite index in a finitely presented group G then H and G are quasi-isometric.

3 (i) Give a brief description of algorithms and algorithmic problems referring to the word problem associated with a finite presentation $\langle A \mid R\rangle$.

(ii) Define the Dehn function and isoperimetric functions with respect to a finite presentation. Show that a finite presentation has a sub-recursive isoperimetric function if and only if there exists an algorithm that solves the word problem with respect to the presentation.

(iii) Let $\langle A \mid R\rangle$ and $\langle B \mid S\rangle$ be finite presentations of some group G. Write down a quasi-isometry from $C(G, A)$ to $C(G, B)$.

Prove that any two finite presentations of G have equivalent isoperimetric functions. Deduce that any finite presentation of G has solvable word problem.

4 (i) Given a finite presentation $\langle A \mid R\rangle$ of a group G, define the path $\widehat{w}(t)$ in the Cayley graph $C(G, A)$ associated with a word $w \in A^*$. Define the fellow traveller condition for pairs of paths and define combing and a combable group.

(ii) Prove that a combable group with a synchronous combing has a finite presentation.

(iii) Let G be a combable group with finite presentation $\langle A \mid R\rangle$. Describe the construction of automata that accept the pairs of words (w_1, w_2) such that $w_1^{-1}w_2 = a$ for all $a \in A$. Show how the automata are part of an algorithm to construct any finite region of the Cayley graph $C(G, A)$. Show that any finite presentation $\langle A \mid R\rangle$ of a combable groups has solvable word problem.

5 (i) Define the growth function associated with a finite presentation $G = \langle A|R \rangle$. Prove that the growth function defined up to $\sim_e$ equivalence of functions is an invariant of quasi-isometry.

Let G be a finitely generated group and A a finite set of generators for G. Prove that the growth function with respect to a presentation $\langle A \,|\, R \rangle$ of G has an exponential upper bound.

(ii) Prove that $\mathbb{Z}^n$ and $\mathbb{Z}^m$ are quasi-isometric if and only if $n = m$.

(iii) Determine the growth function for the finite presentation

$$\langle x, y \,|\, x^{-1}yx = y^3 \rangle.$$