UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part III

Thursday 5 June 2003 9 to 12

## PAPER 49

## THE STANDARD MODEL

*Attempt* **THREE** *questions.*

*There are* **four** *questions in total.*

*The questions carry equal weight.*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** An early proposal for the Standard Model was a theory invariant under $O(3)$ gauge transformations, explicitly with the gauge-scalar particle sector defined by the Lagrangian where a three-component "triplet" gauge field $\mathbf{A}_\mu$ is coupled to a real triplet scalar field $\phi$ as below,

$$\mathcal{L} = -\tfrac{1}{4}\,\mathbf{F}^{\mu\nu}\cdot\mathbf{F}_{\mu\nu} + \tfrac{1}{2}(D^\mu\phi)\cdot D_\mu\phi - \tfrac{1}{8}\lambda\big(\phi^2 - v^2\big)^2,$$
$$\mathbf{F}_{\mu\nu} = \partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu + e\,\mathbf{A}_\mu \times \mathbf{A}_\nu\,, \quad D_\mu\phi = \partial_\mu\phi + e\,\mathbf{A}_\mu \times \phi\,.$$

Show that in this theory the $O(3)$ symmetry is broken by a choice of ground state to $O(2)$. Rewrite the theory in terms of physical fields and determine the masses of the three gauge bosons.

In such a theory when coupling to leptons one has a charged current and its hermitian conjugate and a neutral current. In order for the charges for these to form a closed algebra one must add additional leptons beyond the electron and neutrino in the first family. It is necessary to have both a left-handed and right-handed $O(3)$ triplet

$$L^3 = \tfrac{1}{2}(1-\gamma^5)\begin{pmatrix} E^+ \\ \nu_e\cos\alpha + N\sin\alpha \\ e \end{pmatrix} \qquad R^3 = \tfrac{1}{2}(1+\gamma^5)\begin{pmatrix} E^+ \\ N \\ e \end{pmatrix},$$

where $E^+$ is a heavy positively charged lepton and $N$ a heavy neutral lepton. The kinetic terms are defined by

$$\bar{L}^3(i\gamma\cdot\partial)L^3 + \bar{R}^3(i\gamma\cdot\partial)R^3 + \bar{L}^S(i\gamma\cdot\partial)L^S,$$

where $L^S$ is a left-handed singlet which is necessary to give the conventional kinetic terms for each of the 4 fields. Find the form of this singlet. Assume the neutrino is massless and left-handed.

The charged current in this theory may be shown to be

$$J^+_\lambda = \bar{E}^+\gamma_\lambda(1-\gamma^5)(\nu_e\cos\alpha + N\sin\alpha) + (\bar{\nu}_e\cos\alpha + \bar{N}\sin\alpha)\gamma_\lambda(1-\gamma^5)e$$
$$+\,\bar{E}^+\gamma_\lambda(1+\gamma^5)N + \bar{N}\gamma_\lambda(1+\gamma^5)e,$$

and $J^-_\lambda = (J^+_\lambda)^\dagger$. If $T^+(t)$ is the conserved charge corresponding to the current $J^+_\lambda(x)$ then show that

$$[T^+(t), T^-(t)] \propto Q(t)$$

where $Q(t)$ is the expected electromagnetic charge. You may wish to use the anti-commutation relationship for fermions of species labelled by $A, B$

$$\{\psi^A_\alpha(\underline{x}, t), \psi^{B\dagger}_\beta(\underline{y}, t)\} = \delta^3(\underline{x} - \underline{y})\delta_{\alpha\beta}\delta^{AB}.$$

**2** The covariant derivative in the electroweak sector of the Standard Model is defined by

$$D^\mu = (\partial^\mu + ig\tfrac{1}{2}\mathbf{A}^\mu(\mathbf{x})\cdot\sigma + iYg'B^\mu(x)),$$

where $g$ is the $SU(2)$ coupling constant $g'$ is the $U(1)_Y$ coupling constant $\sigma_i$ are the Pauli matrices and $Y$ is the hypercharge. Using

$$\begin{pmatrix} A_3^\mu \\ B^\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} Z^\mu \\ A^\mu \end{pmatrix},$$

and $g'/g = \tan\theta_W$, show that the interaction of the $Z$ boson with the fermion fields may be written as

$$\mathcal{L}_{Zf} = -\frac{g}{2\cos\theta_w}\sum_i Z^\mu \bar{\psi}^i\gamma_\mu(c_v^i - c_a^i\gamma^5)\psi^i,$$

where the sum is over fermion species. Using the decomposition of the fermions in one generation into their $SU(2)$ and hypercharge representations find the values of $c_v^i$ and $c_a^i$ for the four types of fermion.

Neglecting fermion masses and working in the $Z$ rest frame show that

$$\sum_{\lambda,s_1,s_2}|M|^2 = \frac{g^2M_Z^2((c_v^i)^2+(c_a^i)^2)}{\cos^2\theta_W}$$

where $M$ is the matrix element for $Z\to\bar{\psi}\psi$, and the sum is over initial polarizations and final spins.

Show that for a single fermion the decay rate for $Z\to\bar{\psi}\psi$ is

$$\Gamma = \frac{1}{48\pi}\frac{g^2}{\cos^2\theta_W}((c_v^i)^2+(c_a^i)^2)M_Z.$$

Assuming that neutrinos in any possible family are effectively massless, what information did the first measurement of the total width, and hence decay rate, of the $Z$ boson provide?

[You may use $\sum_\lambda \epsilon_\mu^*(p,\lambda)\epsilon_\nu(p,\lambda) = -g_{\mu\nu} + \frac{p_\mu p_\nu}{p^2}$,

$$\mathrm{tr}(\gamma_\alpha\gamma_\beta\gamma_\gamma\gamma_\delta) = 4(g_{\alpha\beta}g_{\gamma\delta} + g_{\alpha\delta}g_{\beta\gamma} - g_{\alpha\gamma}g_{\beta\delta}), \qquad \mathrm{tr}(\gamma^5\gamma_\alpha\gamma_\beta\gamma_\gamma\gamma_\delta) = 4i\epsilon_{\alpha\beta\gamma\delta},$$

where $\epsilon_{\alpha\beta\gamma\delta}$ is the totally antisymmetric tensor, and

$$\Gamma = \frac{1}{2M_Z}\int \frac{d^3\mathbf{p_1}}{(2\pi)^3 2E_{\mathbf{p_1}}}\frac{d^3\mathbf{p_2}}{(2\pi)^3 2E_{\mathbf{p_2}}}(2\pi)^4\delta(M_Z - E_{\mathbf{p_1}} - E_{\mathbf{p_2}})\delta^3(\mathbf{p_1}+\mathbf{p_2})|M|^2$$

in the centre-of-mass frame.]

**3** In unitary gauge the scalar doublet in the Standard Model can be written as $\phi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + H(x) \end{pmatrix}$, and has hypercharge $Y = \frac{1}{2}$. Show that the *conjugate* doublet $\phi^c(x) = i\sigma_2\phi(x)^*$ has the appropriate gauge transformation properties to give the up quark a mass via the interaction term

$$\mathcal{L}_{q\phi} = -\sqrt{2}[\bar{L}f^+\phi^c R^+ m + \text{hermitian conjugate}],$$

where $L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$, $R^+ = u_R$ and $f^+$ is an arbitrary coupling. (It will be useful to show that $i\sigma_2\sigma_i^* i\sigma_2 = \sigma_i$, where $\sigma_i$ are the Pauli matrices.)

When we consider three families the constant $f^+$ becomes a matrix and there is a similar matrix with elements $f^-$ for the down-type quarks. The diagonalization of these matrices leads to the weak charged current having the form

$$J^\mu = (\,\bar{u},\quad \bar{c},\quad \bar{t}\,)\,\gamma^\mu(1-\gamma^5)V_{CKM}\begin{pmatrix} d \\ s \\ b \end{pmatrix},$$

where $V_{CKM}$ is a unitary matrix and the quarks are expressed in terms of mass eigenstates. Explain why $V_{CKM}$ contains three independent angles and one complex phase and why for massless neutrinos there is no similar mixing in the charged current interactions.

The current current interaction term in the Lagrangian is

$$\mathcal{L}_{cc} = -\frac{g}{2\sqrt{2}}(J^\mu W_\mu + J^{\mu\dagger}W_\mu^\dagger).$$

Show explicitly that under the combined parity and charge-conjugation transformations this Lagrangian is not invariant.

Briefly justify, (without proof), why, if we have right-handed, and hence massive, neutrinos we can add a "Majorana" mass term

$$\mathcal{L}_M = -m_M[(\bar{\nu}(x)_R)^C\nu(x)_R + \text{hermitian conjugate}]$$

as well as the standard "Dirac" mass term, whereas such a term is not allowed for other fermions.

[Under parity transformations $P$

$$\psi(x) \to \gamma^0\psi(x_P) \qquad \bar{\psi}(x) \to \bar{\psi}(x_P)\gamma_0 \qquad W_\mu(x) \to W^\mu(x_P).$$

Under charge conjugation $C$

$$\psi(x) \to C\bar{\psi}^t(x) \qquad \bar{\psi}(x) \to -\psi^t(x)C^{-1} \qquad W_\mu(x) \to -W_\mu^\dagger(x),$$

where $t$ denotes transpose and $C(\gamma^\mu)^t C^{-1} = -\gamma^\mu$.]

**4** Briefly justify the fact that the renormalized coupling in any quantum field theory $g(\mu^2)$ is a function of the renormalization scale and satisfies a renormalization group equation of the general form

$$\frac{dg}{d\ln\mu^2} = \sum_{n=i}^{\infty} b_n g^n,$$

where $b_n$ are constant coefficients unique to a particular theory.

At lowest order this equation can be written in the form

$$\frac{d\alpha}{d\ln\mu^2} = b_0\alpha^2,$$

where $\alpha = g^2/4\pi$, find the solution subject to boundary conditions $\alpha = \alpha_0$ at $\mu^2 = \mu_0^2$ for $b_0 = \pm\beta_0$, where $\beta_0$ is a positive constant. If $b_0$ is negative show that the solution may be rewritten as

$$\alpha(\mu^2) = \frac{1}{\beta_0 \ln(\mu^2/\Lambda^2)}.$$

$b_0$ is negative for the strong QCD coupling. Explain what happens as $\mu^2 \to \infty$ and for small $\mu^2$, and discuss briefly what results this produces in strong interaction physics. Roughly what size should $\Lambda$ be assuming $\beta_0$ is of order unity?

Discuss the behaviour of the coupling with $\mu^2$ for positive $b_0$. This is the case for the Higgs self-coupling. Using the relationship between this coupling and the Higgs mass explain what implication a very large Higgs mass would have on the reliability of the Standard model for very large $\mu^2$.

In QCD the explicit expression for $\beta_0$ is

$$\beta_0 = \frac{11N - 2n_f}{12\pi},$$

where $N$ is the number of colours and $n_f$ the number of quark flavours, i.e. at a quark mass threshold $m_q$ $n_f \to n_f + 1$. At leading order $\Lambda$ is a function of $n_f$ and is defined so that $\alpha(\mu^2)$ is continuous at a quark mass threshold. Show that if $\Lambda_5$ appears in the definition of the coupling for $\mu^2 > m_b^2$ and $\Lambda_4$ for $\mu^2 < m_b^2$, where $m_b$ is the bottom quark mass, then

$$\Lambda_5 = \Lambda_4 \left(\frac{m_b}{\Lambda_4}\right)^{-2/23}.$$