UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 30 May 2003 9 to 12

PAPER 48

ADVANCED QUANTUM FIELD THEORY

Attempt **THREE** *questions.*

There are **four** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Show that the quantum-mechanical amplitude for a non-relativistic particle to be at position $q(T) = \beta$ given $q(0) = \alpha$ can be expressed:

$$U(\beta, \alpha; T) = \int_{q(0)=\alpha}^{q(T)=\beta} \mathcal{D}q(t)\, e^{iS[q(t)]}$$

where $S[q] = \int_0^T dt\big(\tfrac{1}{2}m\dot{q}^2 - V(q)\big)$ is the classical action and $\hbar = 1$. [Your derivation should include a careful definition of the right-hand side of the formula.]

Now let $V(q) = \frac{1}{2}m\omega^2 q^2$ (the potential for a harmonic oscillator) and set $q(t) = x(t) + \xi(t)$ where $x(t)$ is the solution of the classical equation of motion with $x(0) = \alpha$, $x(T) = \beta$ and $\xi(0) = \xi(T) = 0$. Show that $S[q] = S[x] + S[\xi]$ and, assuming $\mathcal{D}q(t) = \mathcal{D}\xi(t)$ for fixed $x(t)$, deduce that

$$U(\beta, \alpha; T) = F(T)\, e^{iS[x(t)]}$$

where $F(T)$ is independent of α and β. [You need not determine $x(t)$ or $F(T)$.]

2 **(a)** Define the quantum correlation functions in a scalar field theory using functional integrals. How are these quantities expressed in the language of canonical (or operator) quantization? Define the generating functional, Z, for general correlation functions and state (without proof) how this is related to the generating functional, W, for connected correlation functions.

(b) Define the effective action, Γ, for a scalar field theory. Explain what is meant by amputated, one-particle-irreducible correlation functions and how these can be calculated from Γ (no proof is required). From your definitions, relate the amputated, one-particle-irreducible three-point function to connected correlation functions.

(c) State the procedure for extracting a field theory propagator from the quadratic part of an action in the functional approach to quantization. Give a detailed derivation of the propagator $\Delta^{ab}_{\mu\nu}(x)$ for a non-abelian gauge field A^a_μ when a gauge-fixing term $-\frac{1}{2\alpha}(\partial^\mu A^a_\mu)^2$ is added to the standard Yang-Mills lagrangian. Comment on the limit $\alpha \to \infty$.

3 Express the integral

$$I_d(M^2) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + M^2)^2}$$

(which is taken over d-dimensional Euclidean momenta) in terms of a Γ-function. Show that your result has simple poles at $d = 4$ and $d = 6$.

Assuming standard Feynman rules, show how certain one-loop contributions to (i) the four-point function for ϕ^4 theory in $d = 4$, and (ii) the two-point function for ϕ^3 theory in $d = 6$, can be expressed in terms of $I_d(M^2)$. Find the counterterms corresponding to the divergences in cases (i) and (ii) if minimal subtraction is adopted.

[$\Gamma(z) = \int_0^\infty dt\, t^{z-1} e^{-t} = \frac{1}{z} + O(1)$ for suitable z.]

4 **(a)** Let ω_i $(i = 1, 2, \ldots, 2m{-}1, 2m)$ be real Grassmann variables and A_{ij} a $2m{\times}2m$ antisymmetric matrix. State the value of

$$\int d\omega_1 \ldots \int d\omega_{2m} \; \exp(-\tfrac{1}{2}\omega_i A_{ij} \omega_j)$$

up to a factor independent of A, and check your answer when $m = 1$.

Let θ_a and $\bar{\theta}_a$ $(a = 1, 2, \ldots, n)$ be complex Grassmann variables and M_{ab} an arbitrary $n{\times}n$ matrix. State the value of

$$\int d\theta_1 \ldots \int d\theta_n \int d\bar{\theta}_1 \ldots \int d\bar{\theta}_n \; \exp(-\bar{\theta}_a M_{ab} \theta_b)$$

up to a factor independent of M, and check your answer when $n = 2$.

Show that your statements regarding the integrals above are consistent if $m = n$ and M_{ab} is antisymmetric, by setting $\theta_a = \frac{1}{\sqrt{2}}(\omega_a + i\omega_{n+a})$ and $\bar{\theta}_a = \frac{1}{\sqrt{2}}(\omega_a - i\omega_{n+a})$.

(b) Consider the quantization of a non-abelian gauge theory with action $S[A^a_\mu]$. Give a formula for the appropriate functional integral over the gauge fields if the gauge-fixing condition $\mathcal{F}_a(A^a_\mu) = 0$ is to be enforced.

Indicate how a generalization of one of the results in part (a) allows the introduction of Faddeev-Popov ghosts in the gauge theory functional integral. State clearly the nature of the ghost fields and explain why their introduction is useful.

Give an example, with brief explanation, of a gauge-fixing condition $\mathcal{F}_a$ for which the ghosts decouple from the gauge fields. Is there a disadvantage in making this gauge choice? Do the ghosts and gauge field always decouple in an abelian theory?

[You may quote the form of an infinitesimal gauge transformation on A^a_μ.]