UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 2 June 2003 9 to 11

PAPER 46

STATISTICAL FIELD THEORY

Attempt **TWO** *questions.*

There are **three** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Explain briefly what is meant by a phase diagram. Give an example of a three–dimensional phase diagram which contains a tricritical point and describe the nature of the different transitions which may occur.

Give an account of the Landau-Ginzburg theory of phase transitions in the context of a scalar field theory paying particular attention to the following topics:

(a) The idea of an order parameter;

(b) The occurrence of first-order and continuous phase transitions and how their defining properties are explained;

(c) The Maxwell construction;

(d) The features of a tricritical point.

The critical indices in the $\lambda\phi^4$ field theory are defined by

$$M \;\sim\; (-t)^{\beta}\ ,\quad (h=0) \qquad \text{and} \qquad M \;\sim\; h^{1/\delta}\ ,\quad (t=0)\ ,$$

where $t = (T - T_C)/T_C$, and h is the applied magnetic field. Calculate β and δ for both an ordinary and a tricritical point.

2 The Ising model in D dimensions is defined on a cubic lattice of spacing a with N sites and with spin σ_r on the r-th site. The Hamiltonian is defined in terms of a set of operators $O_i(\{\sigma\})$ by

$$\mathcal{H}(\mathbf{u}, \sigma) \;=\; \sum_i u_i O_i(\{\sigma\}) \;,$$

where the u_i are coupling constants with $\mathbf{u} = (u_1, u_2, \ldots)$ and $\sigma_r \in \{1, -1\}$. In particular, $\mathcal{H}$ contains the term $-h\sum_r \sigma_r$ where h is the magnetic field. The partition function is given by

$$\mathcal{Z}(\mathbf{u}, C, N) \;=\; \sum_\sigma \exp(-\beta\mathcal{H}(\mathbf{u}, \sigma) - \beta NC) \;.$$

Define the two-point correlation function $G(\mathbf{r})$ for the theory and state how the correlation length ξ parametrizes its behaviour as $r \to \infty$.

State how the magnetization M and the magnetic susceptibility χ can be determined from the free energy F, and derive the relation which expresses χ in terms of $G(\mathbf{r})$.

Explain how a renormalization group (RG) transformation may be defined in terms of a blocking kernel which, after p iterations, yields a blocked partition function $\mathcal{Z}(\mathbf{u}_p, C_p, N_p)$ which predicts the same large-scale properties for the system as does $\mathcal{Z}(\mathbf{u}, C, N)$.

Derive the RG equation for the free energy $F(\mathbf{u}_p, C_p)$ and explain how it may be expressed in terms of a singular part $f(\mathbf{u})$ which obeys the RG equation

$$f(\mathbf{u}_0) \;=\; b^{-pD} f(\mathbf{u}_p) \;+\; \sum_{j=0}^{p-1} b^{-jD} g(\mathbf{u}_j) \;, \qquad (*)$$

where the rôle of the function$g(\mathbf{u})$ should be explained.

Explain the idea of a fixed point, a critical surface and a repulsive trajectory in the context of the RG equations, and sketch some typical RG flows near to a critical surface.

Show how the critical exponents characterizing a continuous phase transition may be derived and, in particular, state under what conditions the second term (the inhomogeneous term) on the right-hand-side of $(*)$ may be safely neglected.

In the case that there are two relevant couplings, $t = (T - T_C)/T_C$ and h, derive the scaling hypothesis for the singular part F_s of the free energy:

$$F_s \;=\; |t|^{D/l_t} f_\pm\left(\frac{h}{|t|^{l_h/l_t}}\right) \;,$$

where the difference between the two functions f_+ and f_- and the meanings of l_h, l_t should be explained.

The following critical exponents are defined:

$$\begin{array}{lll}
\xi \sim |t|^{-\nu} & h = 0 & \\
M \sim |t|^{\beta} & h = 0, \quad T < T_C & \\
\chi \sim |t|^{-\gamma} & h = 0 & \\
C_V \sim |t|^{-\alpha} & h = 0 & \text{(the specific heat)} \\
M \sim |h|^{1/\delta} & T = T_C &
\end{array}$$

Derive the relations

$$\beta\delta \;=\; \beta + \gamma\,, \qquad\qquad \alpha \;=\; 2 - D\nu\,.$$

3 A statistical system at temperature T is described by a scalar field theory in D-dimensions whose effective Hamiltonian H is defined by

$$H = \int_{\Lambda^{-1}} d\mathbf{x}\, \mathcal{H}(\Lambda, \phi(\mathbf{x}))$$

$$\mathcal{H}(\Lambda, \phi(\mathbf{x})) = \tfrac{1}{2}\alpha^{-1}(\Lambda, T)(\nabla\phi)^2 + \tfrac{1}{2}m^2(\Lambda, T)\phi^2 + \frac{1}{4!}g(\Lambda, T)\phi^4 + \dots\,,$$

where Λ is the large momentum (small distance) cutoff and $\mathcal{H}$ is the Hamiltonian density. The magnetic field is set to zero. The partition function is

$$\mathcal{Z} \;=\; \int \{d\phi\} e^{-H(\phi)}\,.$$

Why do the coupling constants depend on Λ? Why is it reasonable to associate $m(\Lambda = 0, T)$ with the correlation length?

By giving an example of a blocking strategy explain how a Renormalization Group (RG) strategy for successively integrating out high-momentum modes may be applied to this model.

By making suitable assumptions show how the Landau-Ginzburg theory of phase transitions may be derived using the RG in the context of this model. Why do you expect this derivation to be invalid for low enough dimension?

In the case of a ϕ^4 scalar field theory the two-point function $G(\mathbf{x})$ and its Fourier transform $\tilde{G}(\mathbf{p})$ are defined by

$$G(\mathbf{x}) \;=\; \langle \phi(0)\phi(\mathbf{x})\rangle\,, \qquad \tilde{G}(\mathbf{p}) \;=\; \int \frac{d^Dp}{(2\pi)^D} e^{-i\mathbf{p}\cdot\mathbf{x}} G(\mathbf{x})\,.$$

State what is meant by the truncated two-point function $\tilde{\Gamma}(\mathbf{p})$.

Explain how in perturbation theory $\tilde{\Gamma}(\mathbf{p})$ may be written as

$$\tilde{\Gamma}(\mathbf{p}) \;=\; \tilde{G}_0^{-1}(\mathbf{p}) + \delta m^2 + \Sigma(\mathbf{p})\,,$$

where the meaning of each of the terms in this expression should be clearly derived. You may quote the rules of perturbation theory without derivation.

Hence show to one-loop order that

$$m^2(0, T) \;=\; m^2(\Lambda, T) + \frac{g}{2}\int \frac{d^Dp}{(2\pi)^D} \frac{1}{\mathbf{p}^2 + m^2(0, T)}\,.$$

Show that this result is consistent with the Landau-Ginzburg assumption that $m^2(0, T) \sim (T - T_C)$ only for $D > D_C$, where the value of D_C for an ordinary critical point should be calculated.

Describe briefly how the value of D_C for a tricritical point is calculated?