UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS   Part III

Wednesday 4 June 2003   1.30 to 4.30

PAPER 45

SYMMETRY AND PARTICLE PHYSICS

Attempt **THREE** *questions.*

There are **four** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 What are the electric charges and isospin quantum numbers of the u and d quarks?

What are the spins, charges and isospin quantum numbers of the possible particle states which may be formed from three u, d quarks? What observed baryons, with appropriate spins and isospins, may these particle states be identified with? Describe also the quark, anti-quark meson states formed from the u, d quarks and their corresponding anti-quarks.

If there is an additional quark with isospin 0 and charge $\frac{2}{3}$ or $-\frac{1}{3}$ what additional baryons and mesons may be formed containing one of the new quarks? Determine their isospins, spins and charges.

[No proofs of rules for combining $SU(2)$ representations need be given but they should be clearly stated. Disregard any orbital angular momentum.]

2 Explain why irreducible $SU(3)$ tensors can be restricted to the form $T^{\alpha_1 \ldots \alpha_k}_{\beta_1 \ldots \beta_l}$, for $\alpha_i, \beta_j \in \{1,2,3\}$, where the indices $\alpha_1 \ldots \alpha_k$ and $\beta_1 \ldots \beta_l$ are completely symmetrised and $T^{\alpha_1 \ldots \alpha_k}_{\beta_1 \ldots \beta_l}$ is zero on contraction of any α index with a β index. Show that the number of independent components is

$$\tfrac{1}{4}(k+1)(k+2)(l+1)(l+2) - \tfrac{1}{4}k(k+1)l(l+1) = \tfrac{1}{2}(k+1)(l+1)(k+l+2)\,.$$

Discuss the tensor product of two irreducible tensors T^α_β and S^γ_δ showing that the dimensions add up as required.

If $|B_r\rangle$ are baryon octet states, $r = 1, \ldots, 8$ and V_i is an operator which also transforms as an octet, so that $[F_i, V_j] = i f_{ijk} V_k$ where F_i are the $SU(3)$ charges, explain in outline why previous results would suggest that $\langle B_r | V_i | B_s \rangle$ should depend on two independent parameters for any r, s, i.

3 The Lie algebra of $SU(2)$ can be written in the form

$$[H, E^{\pm}] = \pm 2E^{\pm}\,, \qquad [E^+, E^-] = H\,.$$

Show how a basis for a representation space may be formed by states $\{|n\rangle\}$, which are eigenvectors of H, $H|n\rangle = n|n\rangle$, such that

$$E^-|n\rangle \propto |n{-}2\rangle \quad \text{or} \quad E^-|n\rangle = 0\,,$$

starting from a state $|\bar{n}\rangle$ satisfying $E^+|\bar{n}\rangle = 0$. For $\bar{n} \geq 0$ and $\bar{n}$ an integer show that a finite dimensional space is obtained.

[You may assume

$$E^+E^-|n\rangle = \tfrac{1}{4}(\bar{n}+n)(\bar{n}-n+2)|n\rangle\,,$$

but indicate how it may be proved by induction.]

What are the possible eigenvalues for H in the representation and what is its dimension?

A rank 2 Lie algebra has two commuting elements. How are the roots $\underline{\alpha}$ defined? The simple roots are $\underline{\alpha}_1, \underline{\alpha}_2$. Define the Cartan matrix $[K_{ij}]$ in this case. Show in outline how the Lie algebra can be written in part in the form

$$[H_i, H_j] = 0\,, \quad [E^+{}_i, E^-{}_j] = \delta_{ij}H_j\,, \quad [H_i, E^{\pm}{}_j] = \pm K_{ji}E^{\pm}{}_j\,, \quad \text{no sum on } j\,.$$

Explain briefly how a representation space with a basis $\{|n_1, n_2\rangle\}$ where $H_i|n_1, n_2\rangle = n_i|n_1, n_2\rangle$ may be obtained starting from a state $|\bar{n}_1, \bar{n}_2\rangle$ satisfying $E^+{}_i|\bar{n}_1, \bar{n}_2\rangle = 0$ with $\bar{n}_i$ integers and $\bar{n}_i \geq 0$.

For a particular Lie algebra $\underline{\alpha}_1 = (1, 0)$, $\underline{\alpha}_2 = (-1, 1)$ and the other positive roots are $\underline{\alpha}_1 + \underline{\alpha}_2$, $2\underline{\alpha}_1 + \underline{\alpha}_2$. How do these roots correspond to non-zero commutators of $\{E^+{}_i\}$? Show that for this algebra

$$E^-{}_1|n_1, n_2\rangle \propto |n_1{-}2, n_2{+}1\rangle \quad \text{or} \quad E^-{}_1|n_1, n_2\rangle = 0\,,$$
$$E^-{}_2|n_1, n_2\rangle \propto |n_1{+}2, n_2{-}2\rangle \quad \text{or} \quad E^-{}_2|n_1, n_2\rangle = 0\,.$$

Find the different states $|n_1, n_2\rangle$ in the representation space when $\bar{n}_1 = 2$, $\bar{n}_2 = 0$.

[Issues of degeneracy need not be considered.]

4 For a Lie algebra with a Lie bracket $[T_a, T_b] = T_c\, c^c{}_{ab}$, $a, b, c = 1, \ldots, D$ show that the matrices $(\hat{T}_a)^c{}_b = c^c{}_{ab}$ form a representation of the Lie algebra. Let

$$\kappa_{ab} = \operatorname{tr}\big(\hat{T}_a \hat{T}_b\big)\,.$$

Explain why κ_{ab} is an invariant tensor satisfying

$$\kappa_{db}\, c^d{}_{ca} + \kappa_{ad}\, c^d{}_{cb} = 0\,.$$

Suppose that the Lie algebra has an invariant subalgebra $\{X_i\}$, so that $[T_a, X_i] = X_j\, c^j{}_{ai}$ for any T_a, which is also abelian. Show that then κ_{ab} has an eigenvector with zero eigenvalue.

If κ_{ab} has an inverse κ^{ab} show that for any matrices $\{t_a\}$ satisfying the Lie algebra

$$[t_a, \kappa^{bc}t_b t_c] = 0\,.$$

For $SU(2)$ determine κ_{ab}. What are the expected eigenvalues of $\kappa^{bc}t_b t_c$ in $SU(2)$ irreducible representations?