UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 29 May 2003 9 to 12

PAPER 44

QUANTUM FIELD THEORY

Attempt **THREE** *questions.*

There are **four** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A real scalar field, $\phi(x)$, has a Lagrangian density

$$\mathcal{L}(x) = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 \quad .$$

Explain how to quantise the field theory and show that in the Heisenberg Picture the quantum field $\phi(x)$ satisfies

$$\left(\partial^2 + m^2\right)\phi(x) = 0 \quad .$$

Deduce that the field can be expressed in terms of mode operators $a(p)$ in the form

$$\phi(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_p}\left(a(p)e^{-ip.x} + a^\dagger(p)e^{ip.x}\right) \quad ,$$

where $p = (E_p, \mathbf{p})$ and $E_p = \sqrt{\mathbf{p}^2 + m^2}$. Setting $f_p(x) = e^{-ip.x}$, show that

$$a(p) = i\int d^3\mathbf{x}\left(f_p^*(x)\frac{\partial}{\partial x^0}\phi(x) - \phi(x)\frac{\partial}{\partial x^0}f_p^*(x)\right)$$

and deduce the corresponding result for $a^\dagger(p)$. Hence show that the equal time canonical commutation relations imply

$$\left[a(p), a^\dagger(p')\right] = (2\pi)^3 2E_p\delta(\mathbf{p} - \mathbf{p}') \quad .$$

The Feynman propagator $\Delta_F(x-y)$ is defined by the equation

$$i\Delta_F(x-y) = \langle 0|T(\phi(x)\phi(y))|0\rangle \quad ,$$

where $|0\rangle$ is the vacuum state of the theory and T indicates that the product of the two fields is time ordered. Show that

$$i\Delta_F(x-y) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_p}\left(\theta(x^0-y^0)e^{-ip.(x-y)} + \theta(y^0-x^0)e^{-ip.(y-x)}\right) \quad .$$

Verify that this result can be obtained from the representation

$$i\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4}e^{-ip.(x-y)}\frac{i}{p^2 - m^2 + i\epsilon} \quad .$$

2 The Dirac equation for a particle of mass m is

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0 \quad ,$$

where $\psi(x)$ is the spinor wave function for the particle and the matrices γ^μ are given by

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \quad \gamma^j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}.$$

Here $\mathbf{1}$ is the 2×2 unit matrix and σ_j are the Pauli matrices which satisfy $\sigma_j^2 = \mathbf{1}$ and $\sigma_j \sigma_l = -\sigma_l \sigma_j = i\sigma_m$ where (j, l, m) is a cyclic permutation of $(1, 2, 3)$. Show that $\psi(x)$ satisfies the Klein-Gordon equation for a relativistic particle of mass m .

Given that, for a Lorentz transformation $L^\mu{}_\nu$, there exists a 4×4 matrix $S(L)$ such that

$$S^{-1}(L)\gamma^\mu S(L) = L^\mu{}_\nu \gamma^\nu \qquad (A) \quad ,$$

show that the Dirac equation is invariant under Lorentz transformations.

An infinitesimal Lorentz transformation can be expressed in the form

$$L^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu \quad ,$$

where $\omega^\mu{}_\nu$ is infinitesimal and $\omega_{\mu\nu} = -\omega_{\nu\mu}$. Given that the corresponding form for $S(L)$ is

$$S(L) = 1 - \frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu} \quad ,$$

where

$$\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right] \quad ,$$

verify that equation (A) is satisfied up to terms linear in $\omega^{\mu\nu}$.

Show that the Dirac equation is invariant under the parity transformation

$$\psi(x) \to \psi_P(x) = \gamma^0 \psi(x_P) \quad ,$$

where x_P is the parity reflection of x .

Show also that the Dirac equation is invariant under the charge conjugation transformation

$$\psi(x) \to \psi_C(x) = i\gamma^2 \left(\psi^\dagger(x)\right)^T \quad .$$

Deduce the effect of the Parity transformation on the charge conjugate spinor $\psi_C(x)$. What does this imply for the intrinsic parity of positrons relative to that of electrons?

3 A field theory has three scalar fields, $\phi(x)$, $\psi(x)$, and $\Phi(x)$. The Lagrangian density is

$$\mathcal{L}(x) = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + \frac{1}{2}(\partial\psi)^2 - \frac{1}{2}m^2\psi^2 + \frac{1}{2}(\partial\Phi)^2 - \frac{1}{2}m^2\Phi^2 + \mathcal{L}_I(x) \quad ,$$

where

$$\mathcal{L}_I(x) = -\frac{g}{2}(\phi^2\Phi + \psi^2\Phi)$$

Write down the equations of motion for the three quantum fields in the *interaction picture* and express them in terms of appropriate mode operators. State the commutation relations that should be obeyed by the mode operators.

Show that the operator that evolves the states of the system from the far past to the far future is

$$S = T\exp\left\{i\int d^4x\mathcal{L}_I(x)\right\} \quad .$$

Calculate to $O(g^2)$ the scattering amplitude for the elastic scattering process in which the initial state comprises a ϕ-particle and a ψ-particle with 4-momenta p and q respectively and the final state comprises a ϕ-particle and a ψ-particle with4-momenta p' and q' respectively. You may quote Wick's theorem.

Draw the Feynman diagram appropriate to the calculation.

4 The Lagrangian density for the electromagnetic 4-vector potential $A_\mu(x)$ is

$$\mathcal{L}(x) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad ,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Derive the equation of motion for A_μ . Describe the gauge invariance of the theory and explain its significance. Show that the choice of Lorentz gauge reduces the equation of motion to the standard wave equation

$$\partial^2 A_\mu = 0 \qquad\qquad (B) \quad .$$

Show that equation (B) can be obtained from the Lagrangian density

$$\mathcal{L}(x) = -\frac{1}{2} \partial_\mu A_\nu \partial^\mu A^\nu \qquad\qquad (C) \quad .$$

Starting from the Lagrangian density in equation (C) show that the canonical equal time commutation relations for the quantum field $A_\mu(x)$ are

$$[A_\mu(x), \Pi^\nu(x')] = i\delta_\mu^\nu \delta(\mathbf{x} - \mathbf{x}') \quad ,$$

where

$$\Pi^\nu(x) = -\partial_0 A^\nu(x) \quad .$$

Assuming that equation (B) is the Heisenberg equation of motion for $A_\mu(x)$ show that

$$A_\mu(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega} \left(a_\mu(k) e^{-ik.x} + a^\dagger(k) e^{ik.x} \right) \quad ,$$

where $\omega = |\mathbf{k}|$ and $k = (\omega, \mathbf{k})$. You may assume also that

$$[a_\mu(k), a^\dagger(k')] = -g_{\mu\nu}(2\pi)^3 2\omega \delta(\mathbf{k} - \mathbf{k}') \quad , \qquad\qquad [a(k), a(k')] = [a^\dagger(k), a^\dagger(k')] = 0 \quad .$$

Given that there exists a vacuum state, $|0\rangle$, such that

$$a_\mu(k)|0\rangle = 0 \quad ,$$

explain how the Gupta-Bleuler method incorporates the Lorentz gauge condition. What are the resulting conditions on the state $|k, \epsilon\rangle = \epsilon^\mu a_\mu^\dagger(k)|0\rangle$? What are the allowed solutions for the polarisation vector ϵ^μ ? Among the solutions identify the longitudinal and transverse polarisations. Explain why longitudinal photons correspond to states of zero norm and why the transverse photons represent the true physical states.

The contribution of $O(e^2)$ to the amplitude for the electron-positron annihilation process

$$e^-(p) + e^+(q) \to \gamma(k, \epsilon) + \gamma(k', \epsilon') \quad ,$$

is

$$T = (ie)^2 \bar{v}(q) M u(p) \quad ,$$

where

$$M = \epsilon'.\gamma \frac{\gamma.(p-k) + m}{(p-k)^2 - m^2} \epsilon.\gamma + \epsilon.\gamma \frac{\gamma.(p-k') + m}{(p-k')^2 - m^2} \epsilon'.\gamma \quad ,$$

and where $u(p)$ and $\bar{v}(q)$ are the electron and positron spinor wave functions respectively. Verify that when ϵ is replaced by k the amplitude $T = 0$. What does this suggest about the production of longitudinal photons in physical scattering processes?