UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 5 June 2003 9 to 12

PAPER 4

CLASSICAL GROUPS

Attempt **THREE** *questions.*

There are **five** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 State and prove Witt's Lemma on isometries of spaces with forms.

2 Precisely two of the groups A_8, $PSL_3(4)$ and $PSL_4(2)$ are isomorphic. Which? Justify your answer briefly.

Write about isomorphisms involving finite classical and alternating groups.

3 Write an essay on symplectic groups. Include a derivation of the order formula and a sketch of a proof of simplicity.

Describe some of the maximal parabolic subgroups, in particular the stabilizer of a point and the stabilizer of a maximal totally isotropic subspace.

4 Show that S_6 has a 2-transitive action of degree 10, in which the stabilizer of a point is the normalizer, $S_3 wr S_2$, of a Sylow 3-subgroup.

Let the group G have subgroups H, K. Show that $G = HK$ if and only if K is transitive in the action on the coset space $(G : H)$ of H in G. Deduce that $S_6 = S_5(S_3 wr S_2)$.

Let $G = Sp_{2m}(2)$, keeping invariant a non-degenerate alternating form B on the space $V = V_{2m}(2)$. Show that G acts on the set $Q(B)$ of quadratic forms which polarize to B, with two orbits, $Q^\epsilon(B)$, $\epsilon = \pm$, and is 2-transitive on each of these. Writing $O^\epsilon_{2m}(2)$ for the stabilizer of a form in $Q^\epsilon(B)$, show that $Sp_{2m}(2) = O^+_{2m}(2)O^-_{2m}(2)$, that is, $O^-_{2m}(2)$ is transitive on $Q^+(B)$.

5 A graph Γ is *distance transitive* if whenever x, y and u, v are pairs of vertices with $d(x, y) = d(u, v)$, there exists an automorphism g of the graph with $xg = u$ and $yg = v$. Here $d(x, y)$ is the *distance* from x to y, that is the length of the shortest path from x to y. Let n and k be integers with $n \geqslant 2k$.

Let Γ be the graph with vertices all the k-subsets of the set $\{1, 2, \ldots, n\}$, two connected by an edge if their intersection has size $k-1$. Show that the symmetric group S_n acts distance transitively on Γ.

Now let Γ be the graph with vertices all the k-dimensional subspaces of an n-dimensional vector space $V = V_n(F)$, two joined if they intersect in a subspace of codimension 1. Show that the linear group $GL_n(F)$ acts distance transitively on Γ.

Show finally that the symplectic group $Sp_{2m}(F)$ acts distance transitively on the graph with vertices the maximal totally isotropic subspaces, two joined if they intersect in a subspace of codimension 1. [*Witt's Lemma can be used without proof here.*]