UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 4 June, 2003 1:30 to 4:30

PAPER 37

Mathematics of Operational Research

Attempt **FOUR** *questions.*

There are **six** *questions.*

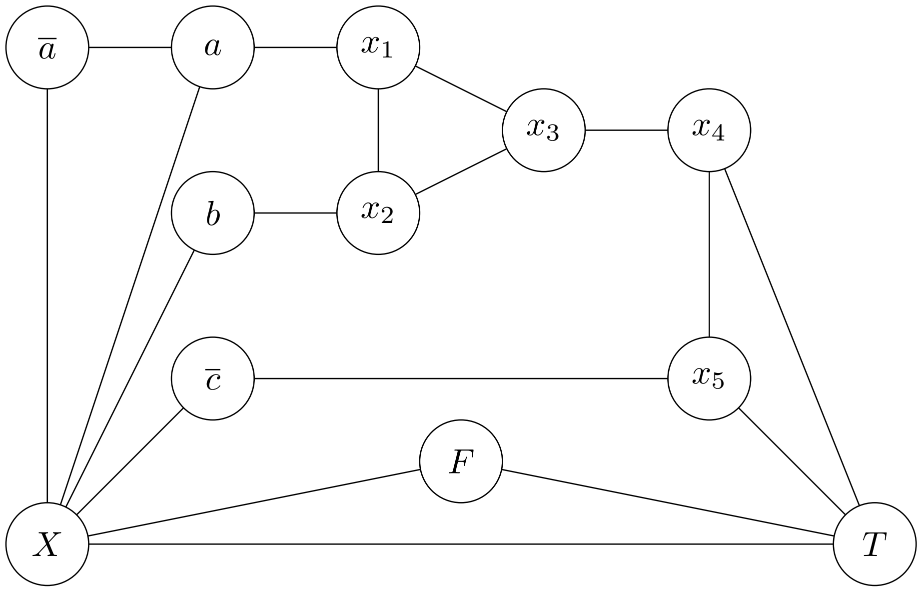
The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Explain the meanings of $\mathcal{NP}$ and $\mathcal{NP}$-complete.

A boolean formula in 3-conjunctive normal form is a conjunction (and) of several clauses, each of which is the disjunction (or) of exactly 3 literals, each of which is either a variable or its negation. An example is '(a or b or $\bar{c}$) and ($\bar{a}$ or b or c)', where $\bar{a}$ denotes the negation of a. In 3-SAT we are given such a formula and asked to say whether there exists an assignment of the variables (to 'true' or 'false') such that the formula is true. Show that 3-SAT is in $\mathcal{NP}$.

In 3-COLOURABILITY we are given a graph as input and asked to decide whether it has a 3-colouring. That is, can we colour the nodes with 3 different colours so that every two nodes that have an edge between them are of different colours? Consider the following statement: *there exists a 3-colouring of the following graph if and only if at least one of the nodes a, b or $\bar{c}$ is coloured the same colour as node T*. Prove the 'only if' part.



Given that the 'if' part is also true and that 3-SAT is $\mathcal{NP}$-complete, show that 3-COLOURABILITY is $\mathcal{NP}$-complete.

2 Explain what is meant by saying that a polyhedron P is full-dimensional.

Let $P = \{x \in \mathbb{R}^n : Ax \geqslant b\}$ and assume that A and b have integer entries which are bounded in absolute value by U. Let

$$\epsilon = \frac{1}{2(n+1)} \left[(n+1)U\right]^{-(n+1)} , \quad P_\epsilon = \{x \in \mathbb{R}^n : Ax \geqslant b - \epsilon e\}$$

where $e^\top = (1, 1, \ldots, 1)$. Show that if P is non-empty, then P_ϵ is full-dimensional.

Give a brief account of the ellipsoidal algorithm for the problem of deciding whether or not P is empty. Describe the inputs to the algorithm and its main steps. You need not derive any detailed formulae, but you should explain enough so that the role of the above result is clear.

3 Employees 1, 2, 3, 4, are to be assigned to the jobs 1, 2, 3, 4, one person per job. The cost of assigning person i to job j is a_{ij}, where these are elements of the matrix

$$\begin{pmatrix} 11 & 12 & 18 & 40 \\ 14 & 15 & 13 & 22 \\ 11 & 17 & 19 & 23 \\ 17 & 14 & 20 & 28 \end{pmatrix}$$

Use the branch and bound method to solve this problem. You should take as a partial solution an assignment of persons $1, \ldots, k$ to k different jobs, $k \leqslant 3$, and use as a lower bound for this partial solution the cost of all the assignments made so far, plus the sum of the least costs with which each of the remaining unassigned jobs could be assigned to one of the persons $k+1, \ldots, 4$ (without requiring each of these jobs to be assigned to a distinct person). Start with the four partial solutions in which person 1 is assigned to job 1, 2, 3 or 4.

Explain how assignment problems can be used with a branch and bound approach to solve the travelling salesman problem.

4 Give an account of Nash's bargaining game, bargaining axioms and arbitration procedure.

Consider the two person non-zero sum game with payoffs

$$\begin{array}{c c} & \begin{array}{cc} II_1 & II_2 \end{array} \\ \begin{array}{c} I_1 \\ I_2 \end{array} & \begin{pmatrix} (2,4) & (8,2) \\ (4,5) & (2,3) \end{pmatrix} \end{array}$$

Find the Nash bargaining solution when the status quo point is taken as the maximin point.

5 Consider an n-person game in which players have strategies $p_1, \ldots, p_n$, each of which may be a mixed strategy. A strategy p_i^* for player i is said to be *dominant* if regardless of what his opponents do it gives him at least as good a payoff as any other strategy he might adopt. Show that if $p_1^*, \ldots, p_n^*$ are dominant strategies for players $1, \ldots, n$ respectively, then $p_1^*, \ldots, p_n^*$ is a Nash equilibrium.

Consider a sealed-bid auction in which the bidders have symmetric independent private values. The winner is the highest bidder and he pays the amount of the second highest bid. Show that a dominant strategy for bidder i is to bid his true valuation.

Suppose, instead, that the winner pays the amount of his own bid. State, or prove the nonexistence of: (a) a dominant strategy for bidder i; (b) a Nash equilibrium.

6 Define the notion of an evolutionary stable strategy (ESS) and derive necessary and sufficient conditions for a strategy $\mathbf{x}^*$ to be evolutionary stable. Use the notation that $e(\mathbf{x}, \mathbf{y})$ is the payoff to a player who uses strategy $\mathbf{x}$ against an opponent who uses strategy $\mathbf{y}$.

Consider the Hawk vs Dove game with payoffs (to the row player) of

$$\begin{array}{cc} & \begin{array}{cc} \text{Hawk} & \text{Dove} \end{array} \\ \begin{array}{c} \text{Hawk} \\ \text{Dove} \end{array} & \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \end{array}$$

Show that the mixed strategy $(\frac{1}{2}, \frac{1}{2})$ is evolutionary stable.