UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 6 June 2003 9 to 12

PAPER 35

QUANTUM INFORMATION THEORY

Attempt **FOUR** *questions.*

There are **five** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $\mathbb{S}$ be an Abelian subgroup of the n-qubit Pauli group $\mathbb{G}_n$. Define a quantum stabilizer code associated with $\mathbb{S}$ and discuss its relation to the symplectic formalism. Give an example of such a code.

If $\mathbb{E}$ is a set of Pauli operators, state and prove the condition under which a stabilizer code is $\mathbb{E}$-correcting.

2 State the Singleton bound separately for classical and quantum codes. Prove the quantum Singleton bound. [*Hint*: You may assume the subadditivity of quantum entropy and the condition for error–correction on a subset of qubits.]

3 What is the capacity of a classical channel? Define a memoryless classical channel and give the formula for the channel capacity.

Classical bits are transmitted through a binary classical channel that preserves the bit with probability $(1-p)$ and makes it unreadable with probability p (a classical erasure channel). Calculate the channel capacity.

4 Define the von Neumann entropy $S(\rho)$ of a density matrix ρ. Prove or disprove the following facts:

(i) $S(\rho)$ is a strictly concave function of ρ:

$$S(b\rho_1 + (1-b)\rho_2) \geq bS(\rho_1) + (1-b)S(\rho_2),$$

where $0 \leq b \leq 1$ and ρ_1, ρ_2 are density matrices (acting in the same Hilbert space), with equality if and only if $b = 0$ or $\rho_1 = \rho_2$.

(ii) $S(\rho)$ decreases when you pass to a subsystem, i.e.,

$$\max\{S(\rho_1), S(\rho_2)\} \leq S(\rho)$$

for ρ acting in a tensor product Hilbert space $\mathcal{K}_1 \otimes \mathcal{K}_2$ and $\rho_1 = \mathrm{tr}_{\mathcal{K}_2}\rho$, $\rho_2 = \mathrm{tr}_{\mathcal{K}_1}\rho$.

(iii) $S(\rho_1) + S(\rho_2) \geq S(\rho)$, where ρ, ρ_1 and ρ_2 are as in (ii).

5 Define the entanglement-assisted capacity of a quantum memoryless channel. State the formula for the entanglement-assisted capacity. Define the binary quantum erasure channel and calculate its entanglement-assisted capacity.