UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 5 June 2003 1.30 to 4.30

PAPER 30

INTERACTING PARTICLE SYSTEMS

Attempt **FOUR** *questions.*

There are **six** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Let $G = (V, E)$ be a finite connected graph, thought of as an electrical network. With each $e \in E$ is associated a number $r_e \in (0, \infty)$ called the (electrical) resistance of e. Let $x, y \in V$, $x \neq y$. Define a *unit flow* on G from x to y, and define the *effective resistance* $R(x, y)$ of G, viewed as a network with source x and sink y.

State the Thomson variational principle in this context, and use it to show the Rayleigh theorem, namely that $R(x, y)$ is a non-decreasing function of the vector $(r_e : e \in E)$.

(b) A particle performs an asymmetric random walk on G. When at vertex u it moves to the neighbour v with probability $p_{u,v}$ proportional to r_e^{-1}, where e is the edge $\langle u, v \rangle$. The walk starts at x. Show that

$$P(\text{walk returns to } x \text{ before it hits } y) = 1 - \frac{1}{d(x)R(x, y)}$$

where $d(x) = \sum_{\langle x,v \rangle} r_{\langle x,v \rangle}^{-1}$, the sum being over all edges $\langle x, v \rangle$ incident to x. [*You may assume any necessary facts about electrical networks and harmonic functions.*]

Explain how this result may be used to show that, for a symmetric random walk on an infinite connected graph I, the deletion of an edge from I makes the process 'no less recurrent' than before.

2 Define a *self-avoiding walk* on a graph G. Let $d \geqslant 2$, and let κ_n be the number of self-avoiding walks of length n on $\mathbb{Z}^d$ starting at the origin. Using the result of question 4(a) or otherwise, prove the existence of the limit $\mu = \lim_{n \to \infty} \kappa_n^{1/n}$, and show that $d \leqslant \mu \leqslant 2d - 1$.

Consider bond percolation on $\mathbb{Z}^d$ with edge-probability p. Define the critical probability p_c, and show that $p_c \geqslant \mu^{-1}$, where μ is the connective constant of $\mathbb{Z}^d$.

Show when $d = 2$ that $p_c \leqslant 1 - \mu^{-1}$.

[*Any general results may be used without proof but should be clearly stated.*]

3 (a) Let E be a finite set, and let $\Omega_E = \{0,1\}^E$. Define an *increasing event* of Ω_E. A probability measure μ on Ω_E is called *positively associated* if $\mu(A \cap B) \geqslant \mu(A)\mu(B)$ for all increasing events A and B. The FKG theorem (or inequality) presents a sufficient condition for the positive association of μ; state this theorem.

(b) Let Λ be the vertex set of a finite box of $\mathbb{Z}^d$, and let $\Omega_\Lambda = \{0,1\}^\Lambda$. The Ising model on Λ may be thought as the probability measure μ_Λ on Ω_Λ given by

$$\mu_\Lambda(\omega) = \frac{1}{Z_\Lambda} \exp\left(\beta \sum_{e=\langle x,y\rangle} \delta_{\omega(x),\omega(y)}\right), \quad \omega \in \Omega_\Lambda,$$

where $\beta > 0$, the summation is over all edges e of Λ, and $\delta_{i,j}$ is the Kronecker delta

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Show that μ_Λ satisfies the condition of the FKG theorem, that is, the sufficient condition for positive association.

(c) Let $\partial\Lambda = \{x \in \Lambda : x \sim y \text{ for some } y \notin \Lambda\}$ be the boundary of Λ. Let $\zeta \in \{0,1\}^{\partial\Lambda}$. Explain what is meant by the Ising model on Λ *with boundary condition* ζ, denoted μ_Λ^ζ. Explain in general terms how the result of part (b) may be used to prove the existence of the infinite-volume limits $\mu^0 = \lim_{\Lambda \to \mathbb{Z}^d} \mu_\Lambda^0$, $\mu^1 = \lim_{\Lambda \to \mathbb{Z}^d} \mu_\Lambda^1$, where the suffices $0, 1$ denote vectors of 0's and 1's respectively.

4 (a) Let $(x_n : n \geqslant 1)$ be a sequence of reals satisfying

$$x_{m+n} \leqslant x_m + x_n \quad \text{for } m, n \geqslant 1.$$

Prove that the limit $\lambda = \lim_{n\to\infty} \{x_n/n\}$ exists and satisfies $\lambda = \inf_m \{x_m/m\}$.

(b) Let $\phi^1_{p,q}$ be the random-cluster measure on $\mathbb{Z}^d$ with $0 < p < 1$ and $q \geqslant 1$, obtained as a weak limit with '1' boundary conditions. Let $e_n = (n, 0, 0, \ldots, 0)$. Show that the limit

$$\alpha(p,q) = \lim_{n\to\infty} \left\{ -\frac{1}{n} \log \phi^1_{p,q}(0 \leftrightarrow e_n) \right\}$$

exists and satisfies

$$\phi^1_{p,q}(0 \leftrightarrow e_n) \leqslant e^{-n\alpha(p,q)} \quad \text{for all } n \geqslant 1.$$

[*Any general results may be used without proof but should be stated clearly.*]

(c) Let $\Lambda_n = [-n, n]^d$ and let $\partial\Lambda_n = \Lambda_n \setminus \Lambda_{n-1}$ be its boundary. Show that there exists $x \in \partial\Lambda_n$ such that

$$\phi^1_{p,q}(0 \leftrightarrow x) \geqslant \frac{1}{|\partial\Lambda_n|} \phi^1_{p,q}(0 \leftrightarrow \partial\Lambda_n)$$

and

$$\phi^1_{p,q}(0 \leftrightarrow e_{2n}) \geqslant \phi^1_{p,q}(0 \leftrightarrow x)^2.$$

Hence prove that

$$-\frac{1}{2n} \log \phi^1_{p,q}(0 \leftrightarrow \partial\Lambda_{2n}) \to \alpha(p,q) \quad \text{as } n \to \infty.$$

5 Write an essay on the topic of the phase transition for the percolation model. Your essay should include definitions of the basic terms used, and should discuss the concepts of critical exponents, universality, and the relevance of conformal functions and stochastic Loewner evolutions in the two dimensional case. The emphasis should be placed on exposition rather than proofs.

6 Explain what is meant by the *voter model* $\xi = (\xi_t : t \geqslant 0)$ on the square lattice $\mathbb{Z}^2$.

Let δ_0 and δ_1 denote the probability measures which place probability 1 on the configuration 'all 0' and 'all 1', respectively, where 0 and 1 are the two states available at each vertex. Show that the set of invariant measures of this voter model is the set of all probability measures of the form $\alpha\delta_0 + (1-\alpha)\delta_1$, $0 \leqslant \alpha \leqslant 1$. [*Any general theorem used should be stated clearly, but the proof may be omitted.*]

Suppose that the initial configuration of the process, at time $t = 0$, has only finitely many vertices in state 1. Show that the limiting distribution of ξ_t, in the limit as $t \to \infty$, is δ_0.