UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 30 May 2003 9 to 12

PAPER 28

GEOMETRY OF 3-DIMENSIONAL MANIFOLDS

Attempt **THREE** *questions.*

There are **five** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let A be a finitely generated free abelian group of rank n, i.e. A is isomorphic to the direct product of n copies of the integers $\mathbb{Z}$. For what values of n is A isomorphic to the fundamental group of a connected, compact, orientable 3-manifold without boundary? How unique is the manifold in each case?

Describe the six flat orientable 3-manifolds.

2 Define the terms nilpotent group and polycylic group. Indicate briefly how to prove that a finitely generated nilpotent group is polycyclic. Show that, up to finite index, a polycyclic PD^3-group is an extension of a normal abelian subgroup of rank 2 by an infinite cyclic group. Under what conditions is this group nilpotent?

If Nil^3 is the nilpotent Lie group consisting of all real matrices of the form

$$X = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

and k is a positive integer, let N_k be the discrete subgroup consisting of all matrices for which x, y and z are integers divisible by k.

Show that $X \mapsto (x \bmod k,\ y \bmod k)$ is the projection map from Nil^3/N_k onto the base T^2 of a locally trivial S^1-fibration. Prove further that

$$H_1(\mathrm{Nil}^3/N_k, \mathbb{Z}) \cong N_k/N_k' \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/k\,.$$

3 Explain the use of the spectral sequence of the group extension

$$1 \to N \to G \to Q \to 1$$

in the proof that, if N is a PD^n-group and Q is a PD^q-group, then G is a PD^{n+q} group.

Suppose now that N is a normal, finitely presented subgroup of the PD^3-group G and that G/N contains elements of infinite order. Show that either N is a PD^2-group, and hence a surface group, or that N is free. In the first case deduce that G contains a subgroup of finite index, which is the fundamental group of a surface fibration over S^1.

4 Outline the main steps in the proof of the Loop Theorem.

Let N be the closure of the complement of a solid torus neighbourhood of a knot K in S^3. If K is not unknotted, show that the inclusion $i : \partial N \to N$ induces an injection $i_* : \pi_1(\partial N, *) \to \pi(N, *)$. (Here $*$ denotes some base point chosen to lie in ∂N.)

Deduce that a knot K is the unknot if and only if $\pi(S^3 - K, *)$ is infitite cyclic.

5 Write an essay on the isometry group $I(M)$ of a Riemannian manifold, paying special attention to dimension three.