UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 6 June 2003 1.30 to 4.30

PAPER 27

MORSE THEORY

Attempt **FOUR** *questions.*

There are **six** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let X be a cell complex, let $X_n \subseteq X$ denote the n-skeleton (the union of cells of dimension no more than n) and $f : S^n \to X$ a continuous map from the n-sphere to X. Show carefully that f is homotopic to a continuous map $g : S^n \to X_n$. You should clearly state any theorems that you use in the process.

2 State and prove the Morse Lemma.

3 Define the notions of Morse function, gradient-like vector field, and the Morse-Smale transversality condition. Explain (without detailed proof) how the latter allows you to compute the integral homology of a manifold, and illustrate it for the height function on the Klein bottle.

4 Let Σ_g be the closed orientable surface of genus g. Prove that any Morse function on $S^1 \times \Sigma_g$ must have at least $4g + 4$ critical points, and describe a Morse function which achieves this minimum.

5 Recall that the Grassmannian $\mathbf{Gr}(k, n)$ of complex k-dimensional subspaces of $\mathbf{C}^n$ is identifiable with the manifold of rank k projection matrices of size $n \times n$. Choose a diagonal matrix $C = diag(c_1, \ldots, c_n)$ with $c_1 < \ldots < c_n$, and consider the function $P \mapsto f(P) = Tr(CP)$. Determine the critical points and their indices, prove that this is a Morse function and hence describe the homology of $Gr(k, n)$. If it helps, you may specialize to concrete values, such as $Gr(2, 4)$, but you may *not* choose $k = 0, 1, n - 1, n$.

[*You may assume that* $\mathbf{Gr}(k, n)$ *is a submanifold of the space of complex* $n \times n$ *matrices, and that its tangent space at* P *is spanned by the infinitesimal conjugation action of unitary matrices.*]

6 Explain what is meant by attaching a handle, with reference to framed links in $\mathbf{R}^3$, and explain how you can construct the four-manifold $S^2 \times S^2$ in this manner.