UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 9 June 2003 1.30 to 4.30

PAPER 25

ADDITIVE AND COMBINATORIAL NUMBER THEORY

Attempt **THREE** *questions.*

There are **six** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $\phi : \mathbb{N} \to \{1, 2, \ldots, r\}$ be an r-colouring of the natural numbers. Prove that there exist integers $a, b > 0$ such that $\phi(a) = \phi(a + b^2)$. [Any intermediate results on which your proof depends should also be proved.]

2 Let $\delta > 0$. Prove that there is a positive integer N such that every subset A of $\{1, 2, \ldots, N\}$ of size at least δN contains an arithmetic progression of length 3. Explain briefly how your argument could be modified to find inside A a triple of the form $\{a, a + d, a + 3d\}$ (with d possibly negative).

3 State and prove Weyl's inequality in the special case of the sum $\sum_{x=0}^{n} e(\alpha x^2)$. [Any lemmas you might need should also be proved.]

4 (i) What is a Plünnecke graph? Prove that if G is a Plünnecke graph with layers $V_0, \ldots, V_n$, and if $D_n(G) \geqslant 1$, then $D_k(G) \geqslant 1$ for every $k \leqslant n$. [You may assume Menger's theorem.]

(ii) State Plünnecke's inequality and indicate briefly how it follows from the result you have just proved.

(iii) Deduce from Plünnecke's inequality that if A is a subset of an Abelian group, $|A + A| \leqslant C|A|$ and k, l are positive integers, then $|kA - lA| \leqslant C^{k+l}|A|$.

5 State and prove Freiman's theorem. [You may assume Plünnecke's inequality, results from the geometry of numbers and basic facts about Freiman isomorphisms.]

6 (i) Let $A \subset \{1, 2, \ldots, N\}$. What does is mean to say that A is *α-quadratically uniform*?

Let $\delta > 0$ and let N be sufficiently large. Prove that if A is an α-quadratically uniform subset of $\{1, \ldots, N\}$ of size δN, for $\alpha = \alpha(\delta)$ small enough, then A contains an arithmetic progression of length four.

(ii) Give an outline of a proof of Szemerédi's theorem for progressions of length four. You should explain what the main steps are, but need not prove them unless you judge that it helps your exposition.