UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS **Part III**

Friday 6 June 2003 9 to 11

PAPER 22

ANALYTIC NUMBER THEORY

Attempt **TWO** *questions.*

There are **five** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Prove by the method of Tchebychev that $\pi(x) = O(x/\log x)$. Hence establish Mertens result that

$$\sum_{p \le x} \frac{\log p}{p} = \log x + O(1)\,.$$

Deduce by the partial summation formula, or otherwise, that, for $\lambda > 1$,

$$\sum_{p \le x} \frac{(\log p)^\lambda}{p} = \frac{1}{\lambda}(\log x)^\lambda + O((\log x)^{\lambda-1})\,.$$

2 Prove that $\zeta(s) \neq 0$ on the line $\sigma = 1$.

Write down a relation between

$$\int_0^x \psi(u)du$$

and $\zeta'(s)/\zeta(s)$, where ψ is the Tchebychev function. Describe briefly how it enables one to verify that the integral is asymptotic to $\frac{1}{2}x^2$ as $x \to \infty$.

What does this imply about the asymptotic value of the nth prime p_n as $n \to \infty$?

3 State and prove the functional equation for $\zeta(s)$.

State also the Riemann-von-Mangoldt formula. Deduce from the latter that if $\gamma_1, \gamma_2, \ldots$ is the increasing sequence of positive ordinates of the zeros of $\zeta(s)$ then $\gamma_n \sim 2\pi n/\log n$ as $n \to \infty$.

4 Describe the main ideas of the Selberg upper-bound sieve. Show how it leads to the result that, for any $\varepsilon > 0$, there exists $x_0 = x_0(\varepsilon)$ such that, for all $a > 0$ and all $x > x_0$,

$$\pi(x+a) - \pi(a) < (2+\varepsilon)x/\log x\,.$$

5 Prove Dirichlet's theorem on primes in arithmetical progressions assuming that $L(1,\chi) \neq 0$ for a real, non-principal character χ.

Let $\pi(x,q,a)$ denote the number of primes $p \le x$ in the arithmetical progression $a, a+q, a+2q, \ldots$ with $(a,q) = 1$. Show that, for $s > 1$,

$$\sum_{p \equiv a (\mathrm{mod}\, q)} \frac{1}{p^s} = s\int_1^\infty \frac{\pi(x,q,a)}{x^{s+1}}dx\,.$$