UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 4 June 2003 9 to 12

PAPER 20

CONSTRUCTIVE TOPOLOGY

Attempt **THREE** *questions.*

There are **five** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Explain what is meant by the terms *nucleus*, *open nucleus* and *closed nucleus* on a frame A. Assuming the result that the nuclei on A form a frame $N(A)$ under pointwise ordering, show that the open and closed nuclei $o(a)$ and $c(a)$ are complementary elements of this frame, and that the set of all open or closed nuclei generates $N(A)$ as a frame.

Now suppose given a frame homomorphism $h\colon B \to A$. We say two nuclei j and k on A are *B-equivalent* if $j(h(b)) = k(h(b))$ for all $b \in B$. By considering the nucleus $\bigvee\{o(h(b)) \wedge c(j(h(b))) \mid b \in B\}$ (where j is an arbitrary nucleus), show that each B-equivalence class has a least element. Show also that these least elements form a subframe $N_B(A)$ of $N(A)$, generated by the elements $\{c(a) \mid a \in A\} \cup \{o(h(b)) \mid b \in B\}$. Deduce that $c\colon A \to N_B(A)$ is a frame homomorphism, and that it is the pushout of $c\colon B \to N(B)$ along h in **Frm**.

[*You may assume the results that $o(a)$ is the least nucleus sending a to 1, and $c(a)$ is the least sending 0 to a.*]

2 Explain what is meant by a *preframe*, and define the tensor product of two preframes. Assuming the result that the category **PFrm** of preframes is symmetric monoidal closed, show that the category **Frm** is coreflective in the category **CMon**(**PFrm**) of commutative monoids in **PFrm**, and show how this result may be used to give descriptions of binary coproducts and filtered colimits in **Frm** in terms of preframes. Hence prove the Tychonoff theorem that an arbitrary product of compact locales is compact.

3 Define the terms *Hausdorff* and T_U for locales, and show that every Hausdorff locale is T_U. Give an example of a space which is Hausdorff as a space, but not T_U as a locale.

An open locale X is said to be *totally connected* if every positive open sublocale of X is connected. Show that the collection of all positive opens in a totally connected locale is a completely prime filter, and that the corresponding point $p\colon 1 \to X$ is a dense sublocale of X. Conversely, if X is open and has a point which is a dense sublocale, show that it is totally connected. Show also that any continuous map from a totally connected locale X to a T_U locale Y is constant (i.e., factors through $X \to 1$). [*Hint: show that the composite $px\colon X \to 1 \to X$ satisfies $1_X \leqslant px$.*]

[*If you wish, you may assume classical logic when answering this question, but full credit will be given only for constructively valid answers.*]

4 Show that the following conditions on a locale X are equivalent:

(i) The closure of every open sublocale of X is clopen (i.e. simultaneously open and closed).

(ii) Every regular open sublocale of X (i.e. every sublocale which is the interior of its closure) is clopen.

(iii) The mapping $\neg\neg\colon \mathcal{O}(X) \to \mathcal{O}(X)$ preserves binary joins.

(iv) The Boolean part X_b of X is a flat sublocale of X.

A locale with these properties is said to be *extremally disconnected*. Show also that if $\mathcal{O}(X)$ is the frame $\mathrm{Idl}(B)$ of ideals of a Boolean algebra B, then X is extremally disconnected iff B is complete.

Now let X be a compact regular locale, and consider the locale γX corresponding to the frame $\mathrm{Idl}(\mathcal{O}(X)_{\neg\neg})$ of ideals of the Boolean algebra of regular open sublocales of X. Show that γX is extremally disconnected, and there is a surjection $\gamma X \to X$. [*Joyal's Lemma may be assumed.*]

5 Define a *propositional geometric theory*, and explain how any such theory $\mathbb{T}$ generates a frame $\mathcal{O}(X_{\mathbb{T}})$ such that the category $\mathbf{Sh}(X_{\mathbb{T}})$ contains a canonical model of $\mathbb{T}$.

Given a commutative ring R (with 1), consider the geometric theory $\mathbb{T}_R$ of 'prime filters' of R, whose primitive propositions have the form $(r \in \mathcal{F})$, $r \in R$, and whose axioms are $(\top \vdash (1 \in \mathcal{F}))$,

$$((r \in \mathcal{F}) \wedge (s \in \mathcal{F}) \vdash (rs \in \mathcal{F}))\ ,$$

$((rs \in \mathcal{F}) \vdash (r \in \mathcal{F}))$, $((0 \in \mathcal{F}) \vdash \bot)$ and

$$(((r+s) \in \mathcal{F}) \vdash (r \in \mathcal{F}) \vee (s \in \mathcal{F}))$$

for all $r, s \in R$. (Thus, classically, a model of $\mathbb{T}_R$ in **Set** is the complement of a prime ideal of R.) Show that any ring homomorphism $R \to S$ induces a locale map $X_{\mathbb{T}_S} \to X_{\mathbb{T}_R}$. If U_r denotes the open sublocale of $X_{\mathbb{T}_R}$ corresponding to the primitive proposition $(r \in \mathcal{F})$, show that $U_r \cup U_s$ is the whole of $X_{\mathbb{T}_R}$ iff there exist $a, b \in R$ such that $ar + bs = 1$. [*Hint: for the necessity of this condition, consider the homomorphism $R \to S$, where S is the quotient of R by the ideal generated by r and s. You may assume that $X_{\mathbb{T}_S}$ is degenerate iff $0 = 1$ in S.*]

Hence show that there is a sheaf $\widetilde{R}$ on $X_{\mathbb{T}_R}$ whose elements of extent U_r, for each r, are the elements of the ring of fractions $R[r^{-1}]$ (that is, equivalence classes of formal fractions a/r^n where $a \in R$ and $n \geqslant 0$, where we identify a/r^n and b/r^m if there exists $k \geqslant 0$ such that $ar^{m+k} = br^{n+k}$). [*You may assume that it suffices to verify the sheaf axiom for coverings of the whole of $X_{\mathbb{T}_R}$ by a pair of basic opens.*]