UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 3 June 2003 1.30 to 4.30

PAPER 2

NOETHERIAN ALGEBRAS

Attempt **THREE** *questions.*

There are **five** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Show that a commutative Artinian ring is Noetherian.

State and prove Wedderburn's theorem concerning the structure of a (not necessarily commutative) semisimple right Artinian ring.

2 Let R be a commutative Noetherian ring. Show that the power series $R[[X]]$ is also Noetherian.

Let I be an ideal of R, and let M and N be R-modules. Write $\hat{R}, \hat{M}$ and $\hat{N}$ for the completions with respect to the I-adic topology. Show that if $M \to N$ is a surjective R-module homomorphism then the induced map $\hat{M} \to \hat{N}$ is also surjective.

By showing that $\hat{R}$ is the image of some suitable power series ring $R[[X_1, \ldots, X_n]]$ under a ring homomorphism, deduce that $\hat{R}$ is Noetherian.

3 Define the set of regular elements of a ring R.

Let A_n be the nth Weyl algebra over $\mathbb{C}$.

Let

$$\begin{aligned} R_1 = &\quad \left\{ \begin{pmatrix} r & s \\ 0 & t \end{pmatrix} : r, s, t \in A_n \right\} \\ R_2 = &\quad \left\{ \begin{pmatrix} r & s \\ t & u \end{pmatrix} : r, s, t, u \in A_n \right\} \end{aligned}$$

Describe the ideals in R_1 and R_2.

In each case, what is the Jacobson radical and the set of regular elements?

Do R_1 and R_2 have left classical ring of quotients?

(You may assume that A_n is left and right Noetherian, and if $rs = 0$ for $r, s \in A_n$ then either $r = 0$ or $s = 0$. You should state and sketch the proof of any other results you use.)

4 State and prove the Hilbert-Serre theorem concerning the Poincaré series of a finitely generated graded module over a commutative positively graded ring.

Define the dimension of a finitely generated left A_n-module, where A_n is the nth Weyl algebra over $\mathbb{C}$. Show that the dimension of A_n/A_nD is $2n-1$ when D is a non-constant element of A_n.

5 Define the ring of differential operators of a commutative ring R. What is the order of a differential operator?

Show that the nth Weyl algebra A_n is the ring of differential operators of $R = \mathbb{C}[X_1, \ldots, X_n]$.

Let $D^i(R)$ be the set of differential operators of order at most i.

Describe the graded ring associated with the filtration $\{D^i(R)\}$ of A_n.

Show how one may define a Poisson bracket on this graded ring by considering the Rees ring associated with this filtration.