UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 30 May 2003 1.30 to 4.30

PAPER 18

CATEGORY THEORY

Attempt **SIX** *questions.*

There are **ten** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $\mathbb{C}$ be a small category.

(i) State and prove the Yoneda Lemma

(ii) Define the Yoneda embedding $H_\bullet$ and show that it is full and faithful.

(iii) Show that $H_\bullet(f)$ is monic if and only if f is monic.

(iv) Show that $H_\bullet(f)$ is epic if and only if f is split epic. [*Recall: a split epic is a morphism e such that $eg = 1$ for some morphism g.*]

2 Suppose that $Y \times Y$ is the product of two copies of Y with projections $p, q : Y \times Y \longrightarrow Y$. Define the diagonal $d = d_Y : Y \longrightarrow Y \times Y$ to be the map $(1_Y, 1_Y)$, so that $pd = qd = 1_Y$.

(i) Show that d is an equalizer of p and q.

(ii) Suppose that $f, g : X \longrightarrow Y$. Show that we can obtain an equalizer of f and g by pulling d_Y back along $(f, g) : X \longrightarrow Y \times Y$.

(ii) Show that a category with terminal object and pullbacks has all finite limits.

3 (i) Give a definition of limits in terms of representability.

(ii) Suppose that $F : \mathbb{I} \times \mathbb{J} \longrightarrow \mathcal{D}$ is such that the functors $F_J = (-, J) : \mathbb{I} \longrightarrow \mathcal{D}$ have limits in D for all $J \in \mathbb{J}$. Show that the assignment

$$J \longmapsto \int_I F(I, J)$$

extends to a functor

$$\int_I F(I, -) : \mathbb{J} \longrightarrow \mathcal{D}$$

and explain in what sense this functor is unique. (Standard facts about representability may be assumed.)

(iii) Suppose in addition that the functor $\int_I F(I, -)$ has a limit. Show that $F : \mathbb{I} \times \mathbb{J} \longrightarrow \mathcal{D}$ has a limit.

(iv) State precisely a Fubini Theorem to the effect that limits commute with limits, and prove it.

4 (i) Explain the notions of ends and coends.

(ii) Prove the Density Formula

$$X(U) \cong \int^{W} \mathbb{C}(U,W) \times X(W)$$

for a presheaf $X \in [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$.

(iii) Deduce that every presheaf is a colimit of representables.

5 Let $\mathcal{C} \underset{G}{\overset{F}{\underset{\perp}{\rightleftarrows}}} \mathcal{D}$ be an adjunction.

(i) Define the *unit* η and the *counit* ε of the adjunction.

(ii) Prove the triangle identities for η and ε.

(iii) Prove that, given functors $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ and natural transformations $1 \xrightarrow{\eta} GF$, $FG \xrightarrow{\varepsilon} 1$ satisfying the triangle identities, there is a unique adjunction between F and G with η as its unit and ε as its counit.

6 (i) Let $\mathbb{A}$ be the category with two objects 0 and 1 and one non-identity map $0 \longrightarrow 1$. Let

$$\Gamma : [\mathbb{A}^{\mathrm{op}}, \mathbf{Set}] \longrightarrow \mathbf{Set}$$

be the functor assigning to a presheaf X the set $X(1)$. Exhibit a chain of adjoints

$$\Pi \dashv \Delta \dashv \Gamma \dashv \nabla.$$

Does Π have a left adjoint? Does ∇ have a right adjoint? Justify your answers.

(ii) Let $O : \mathbf{Cat} \longrightarrow \mathbf{Set}$ be the functor taking a small category to its set of objects. Exhibit a chain of adjoints

$$C \dashv D \dashv O \dashv I.$$

Does this chain of adjunctions extend further in either direction? Justify your answer.

(*When you define a functor you are only required to describe its effect on objects, and when you show adjointness you are not required to carry out any formal checks of naturality.*)

7 **Either** state and prove the General Adjoint Functor Theorem. (If you wish to appeal to an initial object lemma you should prove it.)

Or state and prove the Special Adjoint Functor Theorem. (You may assume the General Adjoint Functor Theorem.)

8 (i) What is a *monad* on a category? What is the *category of algebras* for a monad? What does it mean for a functor to be *monadic*?

(ii) Show that an adjunction gives rise to a monad, and explain briefly why every monad arises in this way.

(iii) Prove that a monadic functor creates limits.

9 (i) Let $G : \mathcal{D} \longrightarrow \mathcal{C}$ be a functor. What is a G-split coequalizer pair? What does it mean for G to reflect G-split coequalizers?

(ii) Suppose that T is the monad on $\mathcal{C}$ induced by an adjunction $F \dashv G : \mathcal{D} \longrightarrow \mathcal{C}$. Define the comparison functor $K : \mathcal{D} \longrightarrow \mathcal{C}^T$. Show that K is full and faithful if and only if G reflects G-split coequalizers.

10 (i) Define the notion of a *monoidal category.* In what sense is a monoidal category a bicategory?

(ii) Show that **Set** has the structure of a monoidal category. Is this structure the only monoidal structure on **Set**? Justify your answer.