UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 9 June 2003 1.30 to 4.30

PAPER 17

SYMPLECTIC GEOMETRY

Attempt **THREE** *questions.*

There are **six** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let (V, ω) be a symplectic vector space. Define the *symplectic orthogonal complement* $W^\perp$ to a subspace $W \subset V$.

Prove that every symplectic vector space is isomorphic to $(\mathbb{R}^{2n}, \omega_{st} = \sum_{j=1}^{n} dx_j \wedge dy_j)$. By choosing co-ordinates, deduce that if $C^2 \subset V^4$ is a linear symplectic subspace of dimension two inside a four-dimensional symplectic vector space and $A : C^\perp \to C$ is a linear map, then the graph of A is a symplectic subspace of V if and only if the determinant $\det(\mathrm{A}) > -1$.

Hence give an example in which the pair $\{C, \mathrm{graph}(A)\}$ are not the image of any pair of complex lines in $\mathbb{C}^2 \cong \mathbb{R}^4$ by an element of $Sp_4(\mathbb{R})$.

2 What is a *symplectic* manifold, and what is a *Lagrangian submanifold* of a symplectic manifold?

Prove that for any smooth manifold, the cotangent bundle T^*M is symplectic and the zero-section is a Lagrangian submanifold. Show moreover that this is the universal model for a symplectic neighbourhood of any Lagrangian submanifold of any symplectic manifold.

Show the graph of a 1-form α, viewed as a subset of T^*M, is Lagrangian if and only if α is closed. Deduce that there is no symplectic six-manifold which contains an open subset which is fibred by Lagrangian three-spheres.

3 When is a group action on a symplectic manifold *Hamiltonian*? Define the *moment map* of such an action.

Suppose the group $\mathbb{S}^1$ acts on (M, ω) in a Hamiltonian fashion and suppose t is a regular value of the moment map μ, and that $\mu^{-1}(t)$ carries a free $\mathbb{S}^1$-action. Show the quotient $\mu^{-1}(t)/\mathbb{S}^1$ admits a natural symplectic structure.

If $L \subset \mu^{-1}(t)/\mathbb{S}^1$ is a Lagrangian submanifold, show that M contains a Lagrangian submanifold which is the total space of a $\mathbb{S}^1$-fibre bundle over L. Taking $M = (\mathbb{C}^{n+1}, \omega_{st})$ and letting $\theta \in \mathbb{S}^1$ act by multiplication by $e^{i\theta}$, deduce that $\mathbb{C}^{n+1}$ contains a Lagrangian submanifold which is a circle bundle over real projective space $\mathbb{RP}^n$.

4 Prove or give counterexamples to each of the following: any auxiliary results used should be given careful statements but need not be proved.

(a) Every symplectic vector bundle $E \to B$ can be written as a direct sum of two Lagrangian subbundles $E \cong L \oplus L'$.

(b) A closed symplectic manifold may only be blown up with fixed weight $\lambda > 0$ finitely many times.

(c) A smooth fibre bundle with two-dimensional base and fibre is always symplectic.

(d) Every complex projective surface admits a Lefschetz pencil of curves of odd genus.

(e) A Kähler 8-manifold cannot have the same Betti numbers as $\mathbb{S}^4 \times \mathbb{T}^4$.

5 What is a *compatible almost complex structure* on a symplectic manifold (X, ω)? Prove that the space of such compatible structures is always non-empty and connected.

The Fubini-Study form Ω on $\mathbb{CP}^2$ is characterised by the fact that its pullback to the unit sphere $\mathbb{S}^5 \subset \mathbb{C}^3$ is the restriction of the standard form from $\mathbb{C}^3$. Deduce from this that (i) a line H in $\mathbb{CP}^2$ has $\int_H \Omega = \pi$ and (ii) the complement of $\{[z_1 : z_2 : z_3] \mid z_3 = 0\} \subset \mathbb{CP}^2$ is symplectomorphic to the ball $(B^4(1), \omega_{st})$.

For any J compatible with Ω and any distinct points p and q in $\mathbb{CP}^2$, there is a J-holomorphic curve of degree one through p and q. Stating carefully any other theorems that you use, deduce that if there is a symplectic embedding $B^4(r_1) \amalg B^4(r_2) \hookrightarrow B^4(1)$ (where all the balls have their standard symplectic structures and $\amalg$ denotes disjoint union) then $r_1^2 + r_2^2 \leqslant 1$.

6 Write an essay on the theory of J-holomorphic curves and their applications (e.g. either to symplectic capacities, or to four-manifolds). You should include some details of some proofs, but may choose which parts of the theory you emphasise.