UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 30 May 2003 1.30 to 4.30

PAPER 14

DIFFERENTIAL GEOMETRY

Attempt **THREE** *questions.*

There are **six** *questions in total.*

The questions carry equal weight.

All the manifolds and related concepts should be assumed to be smooth.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 State the transformation law for the tangent vectors at a point p of a manifold M. Show that the 'velocity vector' $\dot{a}(0)$ of a curve $a(t)$ on M with $\gamma(0) = p$ is a well-defined tangent vector to M.

Let a Lie group G be a subgroup and an embedded submanifold of $GL(n, \mathbb{R})$, for some n. Show carefully that if $\exp(B_1 t)$ and $\exp(B_2 t)$ are in G for all $|t| < \varepsilon$ (where $\varepsilon > 0$) then any linear combination of B_1 and B_2 is the velocity vector of a curve in G at the identity element I. Hence identify the tangent space $T_I G$ with a linear subspace $\mathfrak{g}$ of $n \times n$ matrices. Now suppose that the logarithm maps a neighbourhood of I in G onto a neighbourhood of zero in $\mathfrak{g}$. Prove that $\mathfrak{g}$ is closed under the Lie bracket: if $B_1, B_2 \in \mathfrak{g}$ then $[B_1, B_2] \in \mathfrak{g}$.

[*Standard properties of the logarithm and exponent for matrices may be used without proof, provided these are clearly stated.*]

2 State the defining properties of the exterior derivative. Let U denote the open unit ball in $\mathbb{R}^n$. Using an appropriate vector field and the map $f(x) \mapsto \int_0^1 t^{k-1} f(tx) dt$, or otherwise, construct a linear map $h_k : \Omega^k(U) \to \Omega^{k-1}(U)$, such that $h_{k+1} \circ d + d \circ h_k = \mathrm{id}_{\Omega^k(U)}$, and prove Poincaré Lemma. Show that if α is a differential 1-form on S^2 and $d\alpha = 0$ then $\alpha = df$ for some function f.

State the Hodge decomposition theorem and deduce the relation between harmonic differential forms and de Rham cohomology. Show that the space of harmonic differential forms of top degree on a compact connected oriented manifold M without boundary is one-dimensional.

[*You may assume without proof any properties of the Hodge star operator, provided that you state them accurately.*]

3 Let E be a real vector bundle of rank m over a base manifold B. Prove that E can be given an inner product 'varying smoothly with the fibres'.

Define what is meant by a G-structure on E, where $G \subset GL(m, \mathbb{R})$ is a Lie group. Show carefully that the existence of an $O(m)$-structure on E is equivalent to the existence of an inner product on E. Is it true that every rank m real vector bundle admits an $SO(m)$-structure? Justify your answer.

[*Existence of a partition of unity may be assumed without proof.*]

4 Write an essay on the concept of a connection on a vector bundle over a manifold. Your essay should contain at least three different points of view on connections and a proof of equivalence of at least two of these. (You are *not* expected to prove the existence of connections.) Standard theorems may be used without proof, but should be clearly stated.

5 Define geodesic coordinates on a Riemannian manifold M. Show, stating clearly any preliminary results that you use, that geodesic coordinates exist on a neighbourhood of any point $p \in M$. State and prove Gauss' Lemma.

[*You may assume without proof that the length of* $\dot{\gamma}(t)$ *is constant for any geodesic* $\gamma(t)$.]

6 Define the curvature of a Riemannian metric and the corresponding endomorphism of the tangent bundle $R(X, Y) \in \mathrm{End}(TM)$, where X, Y are two vector fields. Prove the formula $R(X, Y) = D_{[X,Y]} - [D_X, D_Y]$ and deduce the first Bianchi identity $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$ for the Riemann curvature. Construct the Riemann curvature tensor $R_{ij,kl}$ and show that $-R_{ji,kl} = R_{ij,kl} = -R_{ij,lk}$ and $R_{ij,kl} = R_{kl,ij}$.