UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 30 May 2003 9 to 11

PAPER 13

EXTREMAL COMBINATORICS

Attempt **TWO** *questions.*

There are **four** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $\mathcal{A} \subset [n]^{(k)}$. Show that if m is a prime power, and $|A \cap B| \not\equiv k \pmod{m}$ for all distinct A, $B \in \mathcal{A}$, then $|\mathcal{A}| \leq \binom{n}{m-1}$.

Either derive two significant applications of this result,

or show how it might fail if m is not a prime power.

2 State and prove the Ahlswede-Khachatrian theorem, giving the value of $M(n, k, t)$, the maximum size of a t-intersecting family $\mathcal{A} \subset [n]^{(k)}$.

3 (a) Let the number of r-uniform hypergraphs on vertex set $[n]$ that have the hereditary property $\mathcal{P}$ be $d_n^{\binom{n}{r}}$. Prove that d_n decreases with n.

(b) Prove that a strongly $(r+t)$-saturated r-uniform hypergraph of order n has at least $\binom{n}{r} - \binom{n-t}{r}$ edges.

4 (a) Prove that the subset $\mathcal{A}$ of the vertices of the n-cube has vertex boundary at least as large as that of $\mathcal{I}$, the initial segment of the cube order with $|\mathcal{I}| = |\mathcal{A}|$.

(b) State and prove the Four Functions Theorem.

Derive the Harris-Kleitman correlation inequality.

Let $\mathcal{A}_1, \ldots, \mathcal{A}_k$ be intersecting families in $\mathcal{P}[n]$. Prove that

$$\left| \bigcup_{i=1}^{k} \mathcal{A}_i \right| \leq 2^n - 2^{n-k} .$$