UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 3 June 2003 9 to 12

PAPER 1

FINITE DIMENSIONAL LIE ALGEBRAS AND THEIR REPRESENTATIONS

Attempt **QUESTION 1** *and* **THREE** *other questions.*

There are **six** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 a) Let $W = \langle x, y \rangle$ be the standard (defining) representation of $\mathfrak{sl}(2)$. Let

$$U = W \otimes \mathrm{Sym}^3(W).$$

(i) Decompose U into weight spaces, giving a basis for each weight space.

(ii) Draw the weight diagram for U, and identify which irreducible representations occur as submodules of U.

(iii) Give a basis of each irreducible submodule, and indicate highest weight vectors.

b) Now let $V = \langle x, y, z \rangle$ be the standard (defining) representation of $\mathfrak{sl}(3)$, and let $V^* = \langle x^*, y^*, z^* \rangle$ be the dual representation. Let

$$W = V^* \otimes \mathrm{Sym}^2(V)$$

(i) Draw the weight diagram for W and indicate a basis for each weight space.

(ii) Identify the irreducible representations of W and draw the weight diagram for each irreducible representation.

(iii) On their respective weight diagrams indicate a basis for the weight space for each submodule.

2 Let $\mathfrak{h}$ be a Cartan subalgebra of a semi-simple Lie algebra $\mathfrak{g}$.

i) What is meant by the Cartan decomposition of $\mathfrak{g}$? Define all terms, but you may quote results about Cartan subalgebras and semi-simple Lie algebras as needed.

ii) Define the Killing form $B(\ ,\ )$ on $\mathfrak{g}$, and show B is invariant.

iii) Show $B(\ ,\ )$ restricted to $\mathfrak{h}$ is non-degenerate. (You may assume $B(\ ,\ )$ is non-degenerate on $\mathfrak{g}$.)

iv) With respect to a suitable basis of $\mathfrak{sl}(3)$ compute

$$B\left(\begin{pmatrix}1 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & 1\end{pmatrix}\right)$$

3 i) Let $G \subset \mathbb{R}^n$ be a Lie group. Define the tangent space at the identity and describe the adjoint map

$$\mathrm{Ad} : G \to \mathrm{Aut}(T_eG).$$

Explain how the adjoint action of G on T_eG gives rise to a Lie bracket

$$[\ ,\ ] : T_eG \times T_eG \to T_eG.$$

Give this map explicitly.
(Proofs are not required, but you should indicate what needs to be checked.)

ii) Suppose that in $SL(2)$, X is tangent to the curve $H(s) = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$,
and Y is tangent to the curve $g(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$.

Use the definition in i) to calculate $[X, Y]$.
Also compute explicitly $[Y, X]$ demonstrating that

$$[X, Y] = -[Y, X].$$

4 **Prove:** If $\mathfrak{g}$ is semi-simple Lie algebra, V a $\mathfrak{g}$ module, W a $\mathfrak{g}$ submodule, then there exists another submodule W' such that $W \oplus W' = V$.

(You should define any constructions you use, but you may quote standard facts about them.)

5 Let $\mathfrak{g}$ be a semi-simple Lie algebra, with Cartan subalgebra $\mathfrak{h}$.

i) If V is a representation of $\mathfrak{g}$ explain what is meant by the set of weights Λ_V of V.

ii) Explain what is meant by the weight lattice Λ_W, and why $\Lambda_V \subset \Lambda_W$ for all representations V of $\mathfrak{g}$.

iii) Explain what is meant by positive roots of $\mathfrak{g}$.

iv) Explain what the fundamental Weyl chamber W is.

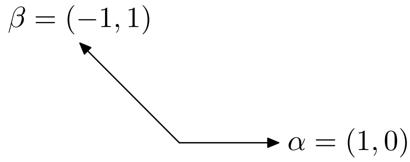
v) If $\mathfrak{g} = \mathfrak{sl}(3)$ with $\mathfrak{h}$ the usual Cartan subalgebra, let $\ell : \mathbb{R}\Lambda_W \to \mathbb{R}$ be given by (not the usual one!)

$$\ell(\mu) = \mu \begin{pmatrix} -\sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1+\sqrt{2} \end{pmatrix}.$$

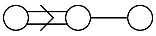
List the positive roots and sketch the fundamental Weyl chamber with respect to the ordering determined by ℓ.

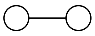
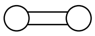
6 i) Define what is meant by an abstract root system.

ii) In the following root system in $\mathbb{R}^2$ the simple roots are drawn



Draw the remaining roots. Explain why each root must be a root and why no other roots can occurs. (You may quote any results you need).

iii) Suppose now that the Dynkin diagram of a three dimensional root system is given by  and has root $\alpha = (1, 0, 0)$, $\beta = (-1, 1, 0)$.

- label the nodes of the Dynkin diagram by α, β appropriately, and let γ be the remaining node;
- give coordinates for γ;
- list all 9 possible positive roots;
- list 3 pairs of positive roots that form a pair of simple roots of a sub-root system of type  ;
- list 3 pairs of positive roots that form a pair of simple roots for a sub-root system of type  .