UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 30 May 2002 9 to 12

PAPER 8

HARMONIC ANALYSIS

Attempt ***THREE*** *questions*

There are ***five*** *questions in total*

The questions carry equal weight

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 What does it mean to say that a sublinear mapping T from a normed space E into $L^0(\mathbf{R}^d)$ is of weak type (E,q)?

State and prove a theorem where a weak type property gives a sufficent condition for convergence almost everywhere.

Suppose that $\nu \in M(\mathbf{R}^d)$, the Banach space of bounded signed measures on $\mathbf{R}^d$. Let $m_u(\nu)(x) = \sup(|\nu|(U)/\lambda(U))$, where the supremum is taken over all open balls U to which x belongs. Show that m_u is of weak type $(M(\mathbf{R}^d), 1)$.

Suppose that $f \in L^1(\mathbf{R})$, and let $F(x) = \int_{-\infty}^{x} f(t)\,dt$. Show that F is differentiable almost everywhere, with derivative f.

2 Suppose that f_1, f_2 and g are non-negative measurable functions on a σ-finite measure space (X, Σ, μ) and that $\int_X g^p\,d\mu < \infty$, where $1 < p < \infty$. Let $1/p + 1/q = 1$.

(i) Suppose that

$$\mu(f_1 > \alpha) < \infty \text{ and } \alpha\mu(f_1 > \alpha) \leqslant \int_{(g>\alpha)} g\,d\mu \text{ for each } \alpha > 0.$$

Show that $\int_X f_1^p\,d\mu \leqslant q \int_X g^p\,d\mu$.

(ii) Suppose that

$$\mu(f_2 > \alpha) < \infty \text{ and } \alpha\mu(f_2 > \alpha) \leqslant \int_{(f_2>\alpha)} g\,d\mu \text{ for each } \alpha > 0.$$

Use (i) to show that $\int_X f_2^p\,d\mu < 2^p q \int_X g^p\,d\mu$.

Show further that $\int_X f_2^p\,d\mu < q^p \int_X g^p\,d\mu$.

(iii) Explain how these results can be applied to non-negative submartingales.

3 Suppose that K is a Calderón-Zygmund singular kernel on $\mathbf{R}^d$: that is, K is a measurable function on $\mathbf{R}^d$ satisfying

(S1) there exists A such that $|K(x)| \leqslant A|x|^{-d}$ for $x \neq 0$;

(S2) there exists B such that $\int_{|x|>2|y|} |K(x-y) - K(x)|\,d\lambda(x) \leqslant B$ for all $y \neq 0$;

(S3) $\int_{r<|x|<R} K(x)\,d\lambda(x) = 0$ for $0 < r < R < \infty$.

Let $K_\epsilon = K\chi_{(|x|>\epsilon)}$. Show that the set of Fourier transforms of the functions K_ϵ is bounded in $L^\infty(\mathbf{R}^d)$. Show that if $f \in L^2(\mathbf{R}^d)$ then $T_\epsilon(f) = K_\epsilon * f$ converges in norm, to $T(f)$, say.

Suppose that ϕ is a bump function on $\mathbf{R}^d$, and that $\phi_\epsilon(x) = \epsilon^{-d}\phi(x/\epsilon)$. Show that $T(\phi_\epsilon) * f \to T(f)$ in norm and almost everywhere, as $\epsilon \to 0$.

[*You may assume standard approximate identity convergence results.*]

4 For $0 < \alpha < d$, let $R^{(\alpha)}(x) = C_\alpha |x|^{\alpha-d}$ be the Riesz potential on $\mathbf{R}^d$, where C_α is chosen so that the Fourier transform is $FR^{(\alpha)}(\xi) = (2\pi|\xi|)^{-\alpha}$. Suppose that $1 < p < d/\alpha$ and that $1/q = 1/p - \alpha/d$. Show that if $f \in L^p(\mathbf{R}^d)$ then

$$I_\alpha(f)(x) = C_\alpha \int_{\mathbf{R}^d} |x-y|^{\alpha-d} f(y)\, dy$$

is defined almost everywhere, and that there exists a constant A such that $\|I_\alpha(f)\|_q \leqslant A\|f\|_p$ for all $f \in L^p(\mathbf{R}^d)$.

[You may quote any Marcinkiewicz interpolation theorem that you need.]

Show that given $1 < p < d/\alpha$ there is no other index q for which such an inequality holds.

Explain how this result can be used to show that the Sobolev space $L_1^p(\mathbf{R}^d)$ is continuously embedded in $L^{q_1}(\mathbf{R}^d)$, where $1/q_1 = 1/p - 1/d$.

5 Let f and $s_1, \ldots, s_d$ be non-negative measurable functions on $\mathbf{R}^d$which satisfy

$$f(x) \leqslant \int_{-\infty}^{\infty} s_j(x_1, \ldots, x_{j-1}, t, x_{j+1}, \ldots, x_d)\, d\lambda(t)$$

for all $x = (x_1, \ldots, x_d) \in \mathbf{R}^d$. Show that

$$\|f\|_{d/(d-1)} \leqslant \left[\prod_{j=1}^{d} \|s_j\|_1\right]^{1/d}.$$

(i) Show that the Sobolev space $L_1^1(\mathbf{R}^d)$ embeds continuously in $L^{d/(d-1)}(\mathbf{R}^d)$.

(ii) K is a compact subset of $\mathbf{R}^d$ and $S_1, \ldots, S_d$ are the images of K under the orthogonal projections onto the d co-ordinate planes. Show that

$$\lambda_d(K)^{d-1} \leqslant \prod_{j=1}^{d} \lambda_{d-1}(S_j),$$

where λ_r is r-dimensional Lebesgue measure.