UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 7 June 2002 1.30 to 4.30

PAPER 74

APPLICATIONS OF DIFFERENTIAL GEOMETRY TO PHYSICS

*Attempt no more than **FOUR** questions*

*There are **seven** questions in total*

The questions carry equal weight

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A six-dimensional Lie group G has Lie algebra $\mathfrak{g}$ with basis $\mathbf{e}_a$, $a = 1, 2, \ldots, 6$. The only non-vanishing brackets are

$$\big[\mathbf{e}_1, \mathbf{e}_2\big] = \mathbf{e}_6,$$

$$\big[\mathbf{e}_1, \mathbf{e}_3\big] = \mathbf{e}_4,$$

$$\big[\mathbf{e}_2, \mathbf{e}_3\big] = \mathbf{e}_5.$$

Show that the symmetric tensors

$$\mathbf{e}_4 \otimes \mathbf{e}_4, \qquad \mathbf{e}_5 \otimes \mathbf{e}_5, \qquad \mathbf{e}_6 \otimes \mathbf{e}_6,$$

and

$$\mathbf{e}_1 \otimes \mathbf{e}_5 + \mathbf{e}_5 \otimes \mathbf{e}_1 + \mathbf{e}_3 \otimes \mathbf{e}_6 + \mathbf{e}_6 \otimes \mathbf{e}_3 - \mathbf{e}_2 \otimes \mathbf{e}_4 - \mathbf{e}_4 \otimes \mathbf{e}_2$$

are invariant under the adjoint action of $\mathfrak{g}$ on itself. Hence construct a four-parameter family of bi-invariant metrics on G.

2 Give a brief outline of the properties of the exterior derivative on differential forms on a manifold M. Explain how given a map $\phi : N \to M$ from another manifold N into M how differential forms on M may be pulled back to N and show that pull-back commutes with exterior differentiation.

Show using this formalism how one writes down an action principle for a $(p+1)$-dimensional submanifold Σ of a n-dimensional spacetime $\{M, g, F\}$ equipped with metric g and closed $p+2$ form field F. Explain why, despite the introduction of a $p+1$ form field A such that $F = dA$, the resulting equations are invariant under gauge transformations of the form $A \to d\Lambda$, for some p-form field Λ.

3 State the Cartan-Maurer relations for the left-invariant one and right-invariant one-forms λ^a, ρ^a, $a = 1, 2, \ldots, \dim G$, on a Lie group G with structure constants $C_a{}^b{}_c$. Give also the Lie brackets of the dual vector fields.

Suppose that G is semi-simple. Show that G admits a bi-invariant metric B_{ab}, and if $C_{abc} = B_{be} C_a{}^e{}_c$, then $C_{abc} = C_{[abc]}$. Hence show that the expression

$$\frac{1}{3!} C_{abc} \lambda^a \wedge \lambda^b \wedge \lambda^c$$

defines a bi-invariant closed three-form on the Lie group G.

4 Define the Killing form of a Lie algebra. Under what conditions is the form non-singular? Give examples when the metric is Riemannian, when it is Lorentzian, and give a non-abelian example in which it degenerates.

Show that every semi-simple Lie group equipped with its Killing form is an Einstein manifold.

Show that every semi-simple Lie group admits a flat metric preserving connection and calculate it's torsion tensor $T_a{}^b{}_c$. What are

$$T_a{}^a{}_c, \qquad \text{and} \qquad T_a{}^b{}_c T_b{}^a{}_d \quad ?$$

5 Define a Poisson manifold and give conditions under which the Poisson bracket satisfies the Jacobi identity. Define a symplectic manifold and show that it is necessarily a Poisson manifold. Is the contrary true? If your answer is no, give a counter-example.

6 A group G acts on a symplectic manifold $\{P, \omega\}$ preserving both the symplectic form ω and a Hamiltonian function H. Define the associated moment map $\mu :\to \mathfrak{g}^\star$ and show how it leads to constants of the motion which commute with the Hamiltonian flow. Give a sufficient condition under which the Poisson algebra of the moment maps and the Lie algebra of G coincide.

Illustrate your answer by calculating the moment maps for an isotropic, i.e. $SO(3)$-invariant, oscillator in three dimensions for which you may take $P = \mathbb{R}^6 \equiv \mathbb{C}^3$, and

$$\omega = \sum dp_i \wedge dq^i = \sum \sqrt{-1}\, dz_i \wedge d\bar{z}_i,$$

$$H = \sum |z_i|^2,$$

with $z_i = \frac{1}{\sqrt{2}}(p_i + \sqrt{-1}\, q_i)$, $i = 1, 2, 3$ and $G = U(3)$.

7 Define a principal fibre bundle with structural group G and show that every such bundle admits a right action of G. Show how to define vector bundles associated to the principal bundle. Illustrate your answer by considering frame bundle of a manifold.

By considering $E(3,1)/SO(3,1)$, show how the Poincare group $E(3,1)$ may be thought of as the bundle of ortho-normal frames of Minkowski spacetime $\mathbb{E}^{3,1}$. Give the analogous construction for de-Sitter and Anti-de-Sitter spacetime.