UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 3 June 2002 9 to 12

PAPER 73

BLACK HOLES

*Attempt **THREE** questions*

*There are **four** questions in total*

The questions carry equal weight

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A scalar field ϕ obeys the Klein-Gordon equation

$$(-\Box + m^2)\phi = 0.$$

In the remote past, and in the distant future, spacetime is static, but is time dependent in between. Explain how to quantise this theory in a static region of spacetime.

Positive frequency modes in the far future, p_{+i}, can be written as a superposition of positive and negative frequency modes in the distant past, p_{-i} and n_{-i} respectively, as

$$p_{+i} = \sum_j p_{-j} A_{ji} + n_{-j} B_{ji}.$$

Derive a similar expression for negative frequency modes in the far future, n_{+i}.

Derive an expression for the expectation value of the number operator for the i^{th} mode in the distant future given that the initial state is the vacuum.

Discuss briefly the application of this method to black hole particle creation.

2 State and prove the black hole area theorem.

Two black holes, B_1 and B_2 of area A_1 and A_2 collide to form a new black hole, B_3 of area A_3. Give an argument to show that

$$A_3 \geq A_1 + A_2.$$

Suppose that two Schwarzschild black holes of equal mass m merge very slowly. In so doing, they will emit copious amounts of radiation in a complicated time dependent fashion. Eventually, they will settle down to form a new Schwarzschild black hole of mass M. Deduce that

$$M \geq \sqrt{2}m.$$

What is the maximum amount of energy in the radiation?

3 In a vacuum asymptotically flat spacetime that admits a complete non-singular spacelike hypersurface Σ, show that the mass must be positive or zero.

Show that if the mass is zero, then there must be a spinor ϵ such that

$$h_i^j \nabla_j \epsilon = 0$$

where h_{ij} is the induced metric on Σ.

Suppose that in addition that the spacetime is static, deduce that there must be a spinor that obeys

$$\nabla_i \epsilon = 0.$$

Give an example of such a spacetime.

4 Describe how to construct the Penrose diagram of the Reissner-Nordstrom metric with $Q^2 = M^2$.

Find by appropriate coordinate transformations, starting with the Reissner--Nordstrom metric in Schwarzschild-type coordinates, the metric in spatially conformally flat coordinates, as

$$ds^2 = -W^2 dt^2 + V^2(dx^2 + dy^2 + dz^2)$$

where W and V are functions of x, y and z which are to be determined.

A clever, but ignorant, physicist asserts that this spacetime does not really contain a black hole because one can travel an infinite proper distance inwards towards the central region in a spacelike fashion without encountering either a trapped surface or a singularity. How would you convince him or her that their views were incorrect and that the black hole was real?