UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 5 June 2002 1.30 to 3.30

PAPER 72

GENERAL RELATIVITY

Attempt ***THREE*** *questions*

There are ***four*** *questions in total*

The questions carry equal weight

The signature is $(+\,-\,-\,-)$, *and the curvature tensor conventions are defined by*

$$R^i{}_{kmn} = \Gamma^i{}_{km,n} - \Gamma^i{}_{kn,m} - \Gamma^i{}_{pm}\Gamma^p{}_{kn} + \Gamma^i{}_{pn}\Gamma^p{}_{km}\,.$$

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Starting from the definition given on the front page prove the *Ricci identity* for a vector field X^i

$$X^i{}_{;kl} - X^i{}_{;lk} = R^i{}_{jkl}X^j.$$

Find the generalization to the commutator of second covariant derivatives of a tensor of arbitrary valence $T^{i\cdots}{}_{n\ldots}$. You may assume the connection is symmetric.

Assume that φ is a scalar field. Show that the second derivatives are symmetric, i.e., $\varphi_{;lm} = \varphi_{;ml}$. Compute (and where possible simplify) the following third derivatives: $\varphi_{;(lm)n}$ and $\varphi_{;l[mn]}$.

Prove that for any valence 2 tensor T^{ik}

$$T^{ik}{}_{;ik} = T^{ik}{}_{;ki}.$$

2 Write an essay on the Pound-Rebka experiment, its implications, the strong principle of equivalence and the derivation of Einstein's field equations for a perfect fluid.

3 A perfect fluid with 4-velocity u^a is described by a number density n, energy density ρ and pressure p, all three being measured in the rest frame of the fluid. The specific enthalpy is $\epsilon = (\rho + p)/n$. Only n and ϵ are independent; $p = p(\epsilon)$ and $\rho = n\epsilon - p$. The velocity is determined from a velocity potential χ via $u_a = \chi_{,a}/\epsilon$. Consider the action

$$S = \int \sqrt{-g}\Big(p(\epsilon) + \frac{n}{2\epsilon}\big[g^{ab}\chi_{,a}\chi_{,b} - \epsilon^2\big]\Big)\, d\Omega,$$

an integral over all of space-time. Use it to establish

$$g_{ab}u^a u^b = 1, \quad \frac{dp}{d\epsilon} = n, \quad \big(nu^a\big)_{;a} = 0,$$

and determine the energy-momentum tensor T^{ab}. Obtain equations governing the evolution of n, ρ and u^a.

Now assume that space-time is almost flat

$$ds^2 = \big(1 + 2\phi(\mathbf{x})\big)\, dt^2 - dx^2 - dy^2 - dz^2,$$

where the Newtonian potential ϕ is t-independent and $|\phi| \ll 1$. Further the fluid motion is slow, $u^a \approx (1, \mathbf{v})$ where $|\mathbf{v}| \ll 1$. Obtain the linearized equations of motion for the fluid and interpret them.

4 Write an essay on horizons in general relativity including definitions and examples.