UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 31 May 2002 9 to 12

PAPER 71

COSMOLOGY

Attempt ***THREE*** *questions*

There are ***seven*** *questions in total*

The questions carry equal weight

You may make free use of the information given on the accompanying sheet.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Explain the origin of the luminosity-distance relation between the measured flux or apparent luminosity F of an object in an expanding universe and its redshift z:

$$F = \frac{L}{4\pi a_0^2 r^2 (1+z)^2}\,,$$

where L is the absolute luminosity, $a_0 = a(t_0)$ is the scalefactor today ($t{=}t_0$) and r is the radial coordinate in the 'astronomers' FRW metric. In particular, define the luminosity distance d_L of the object.

(b) In an open matter-dominated universe ($k < 1, P = 0$), show that the curvature is given by

$$-k = a_0^2 H_0^2 (1 - \Omega_0)\,,$$

where the relative matter density fraction today is $\Omega_0 \equiv \Omega_{\mathrm{M}0} = 8\pi G \rho_{\mathrm{M}}(t_0)/3H_0^2$, with matter density $\rho_{\mathrm{M}}(t)$ and Hubble parameter today H_0.

Determine that the proper distance d_S to an object at radial coordinate r and at the same cosmic time t is given today by

$$d_S = H_0^{-1}(1-\Omega_0)^{-1/2} \sinh^{-1}\left[a_0 H_0 (1-\Omega_0)^{1/2} r\right]\,. \qquad (\dagger)$$

(c) By considering radial photon propagation, show that the proper distance d_S (in the same open matter-dominated universe) can be re-expressed in terms of the redshift z as

$$d_S = a_0 \int_0^z \frac{dz'}{H(z')} = H_0^{-1} \int_0^z \frac{dz'}{(1+z')(1+\Omega_0 z')^{1/2}}\,. \qquad (*)$$

Hence, or otherwise, show that the radial coordinate r is related to the redshift z by

$$r = 2(a_0 H_0)^{-1} \frac{\Omega_0 z + (2-\Omega_0)[1-(1+\Omega_0)^{1/2}]}{\Omega_0^2 (1+z)}\,.$$

Find the limiting value of r at large redshift $z \gg \Omega^{-1}$.

[*Hint: Consider the substitution* $u^2 = (1-\Omega_0)/[\Omega_0(1+z)]$ *and apply the relations* $\sinh(A-B) = \sinh A \cosh B - \cosh A \sinh B$ *and* $\cosh(A-B) = \cosh A \cosh B - \sinh A \sinh B$ *as appropriate.*]

2 Consider a model with a new fermion Y which has a tunable mass m_Y (two spin states $g_Y = 2$ and chemical potential $\mu_Y = 0$). It couples to a gauge boson which acquires its mass through a Higgs mechanism at a GUT-scale temperature $T_{\rm c}$. For $T < T_{\rm c}$, the interaction rate is given by $\Gamma_{\rm int} = G_Y^2 T^5$ with $G_Y = \alpha_Y^2/m_Y^2$ and $\alpha_Y \approx 10^{-3}$.

(a) Find the non-relativistic threshold mass $m_{Y{\rm nr}}$ at which the tunable fermion mass m_Y and the decoupling temperature $T_{\rm D}$ become equal $T_{\rm D} \approx m_Y$. (You may assume that for $T_{\rm D} > 10^3$GeV, we have the effective spin degrees of freedom $\mathcal{N} \approx 10^2$.) Hence, find the "freeze-out" fermion number-to-entropy density ratio for masses $m_Y \ll m_{Y{\rm nr}}$,

$$\frac{n_Y}{s} \approx 10^{-3} \left(\frac{m_{Y{\rm nr}}}{m_Y}\right)^{1/2} \exp\left[-(m_{Y{\rm nr}}/m_Y)^{1/3}\right] .$$

(b) Roughly estimate the baryon number-to-entropy ratio n_B/s given the following information: We live in a flat ($\Omega = 1$) matter-dominated universe, the relative baryon density today is $\Omega_{B0} \approx 0.1$, the age of the universe is $t_0 \approx 10\,{\rm Gyr} \approx 10^{43}{\rm GeV}^{-1}$, the CMB temperature is $T_\gamma = 3K = 10^{-13}$GeV, the baryon mass is $m_B \approx 1\,{\rm GeV}$ and there are three neutrino species (with $T_\nu \approx (4/11)^{1/3} T_\gamma$).

By comparing n_B/s with n_Y/s, estimate the range of masses m_Y for which this new fermion is compatible with the standard cosmology.

(c) A typical massless gauge boson coupling above the GUT-scale critical temperature $T > T_{\rm c}$ is $\Gamma_{\rm int} = \alpha_Y^2 T$. Discuss the maintenance of thermal equilibrium in this case $T > T_{\rm c}$ and its implications for the initial particle distribution of the Y-fermion.

3 Consider a flat ($k = 0$) FRW universe filled with a non-relativistic matter density $\rho_{\rm M}$ (with $P_{\rm M} = 0$) and a relativistic 'dark energy' density $\rho_{\rm Q}$ satisfying an equation of state $P_{\rm Q} = -\rho_{\rm Q}/3$.

(a) Show that the density in 'dark energy' behaves as

$$\rho_{\rm Q} = \frac{3H_0^2(1-\Omega_{\rm M0})}{8\pi G\, a^2}\,,$$

where at present $t = t_0$ we have the Hubble parameter H_0, the fractional density in matter $\Omega_{\rm M0} = 8\pi G\rho_{\rm M}(t_0)/3H_0^2$ and the scalefactor is normalized to unity $a_0 = 1$.

(b) Use the Friedmann and Raychaudhuri equations to show that

$$2\mathcal{H}' + \mathcal{H}^2 - \beta^2 = 0\,,$$

where $\beta \equiv H_0(1-\Omega_{\rm M0})^{1/2}$ and $\mathcal{H}$ is the conformal Hubble parameter $\mathcal{H} = a'/a$ with primes denoting derivatives with respect to conformal time τ ($d\tau = dt/a$).

(c) Hence, or otherwise, find the parametric solution for the scalefactor a and physical time t:

$$a(\tau) = \frac{\Omega_{\rm M0}}{2(1-\Omega_{\rm M0})}\left[\cosh\beta\tau - 1\right],$$

$$t(\tau) = \frac{H_0^{-1}}{2}\frac{\Omega_{\rm M0}}{(1-\Omega_{\rm M0})^{3/2}}\left[\sinh\beta\tau - \beta\tau\right].$$

(d) Find an expression for the age of this universe t_0 in terms of $\Omega_{\rm M0}$ and H_0 and show that it always satisfies

$$\tfrac{2}{3}H_0^{-1} \le t_0 \le H_0^{-1}\,.$$

[*Hint: You may wish to use the expansion* $\sinh^{-1} x = x - \frac{1}{6}x^3 + ...$ *for* $|x| < 1$.]

4 Consider an almost FRW universe with a scalar field ϕ obeying the following evolution equations

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi}{3m_{\rm pl}^2}\left[\tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2}(\nabla\phi)^2 + V(\phi)\right] ,$$
$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} - \frac{1}{a^2}\Delta\phi = -\frac{dV}{d\phi} ,$$

where $V(\phi)$ is the scalar field potential and Δ is the spatial Laplacian.

(a) Discuss the conditions necessary for the universe to enter an extended period of inflationary growth.

(b) Apply these conditions (including the slow-roll approximation) for the exponential potential, $V(\phi) = m_{\rm pl}^4 \exp\left(-A\phi/m_{\rm pl}\right)$, to find the power law inflationary solutions,

$$\phi(t) = \frac{2m_{\rm pl}}{A} \ln\left[\left(\frac{A^4 V_0}{96\pi m_{\rm pl}^2}\right)^{1/2} (t+b)\right] ,$$
$$\frac{a(t)}{a(t_{\rm i})} = \left(\frac{t+b}{t_{\rm i}+b}\right)^{16\pi/A^2} = \exp\left(\frac{8\pi}{A m_{\rm pl}}\left[\phi(t) - \phi(t_{\rm i})\right]\right) , \qquad (\dagger)$$

where $t_{\rm i}$ and b are constants.

(c) Briefly explain why the model ($\dagger$) faces a serious reheating problem. Assume, instead, that an additional mechanism operates to end inflation at $\phi_{\rm R} \approx 30 m_{\rm pl}/A$ and to reheat the universe almost instantaneously; estimate the reheat temperature of the universe $T_{\rm R}$ (assume that the effective spin degrees of freedom $\mathcal{N} \approx 10^3$). In this case what is the minimum initial value $\phi(t_{\rm i})$ required to solve the flatness problem?

5 In a flat FRW universe ($\Omega = 1$), in synchronous gauge (specifying metric perturbations with $h^{0\mu} = 0$), the perturbations of a multicomponent fluid obey the following evolution equations

$$\delta'_N + (1+w_N) i\mathbf{k}\cdot\mathbf{v}_N - \tfrac{1}{2}(1+w_N)h' = 0\,,$$
$$\mathbf{v}'_N + (1-3w_N)\frac{a'}{a}\mathbf{v}_N + \frac{w_N}{1+w_N} i\mathbf{k}\delta_N = 0\,,$$
$$h'' + \frac{a'}{a}h' - 3\left(\frac{a'}{a}\right)^2 \sum_N (1+3w_N)\Omega_N\delta_N = 0\,,$$

where δ_N is the density perturbation, Ω_N is the fractional density, $\mathbf{v}_N$ is the velocity and $P_N = w_N\rho_N$ is the equation of state of the Nth fluid component, and $\mathbf{k}$ is the comoving wavevector ($k = |\mathbf{k}|$), h is the trace of the metric perturbation and primes denote differentiation with respect to conformal time τ ($d\tau = dt/a$).

(a) Consider a universe filled with cold dark matter ($P_C = 0$) and baryons with $P_B = c_s^2\rho_B$ ($c_s \ll 1$) at late times in the matter-dominated era so that $\Omega_C + \Omega_B \approx 1$. By considering a frame comoving with the cold dark matter and by making appropriate approximations, show that the above evolution equations can be reduced to

$$\delta''_C + \frac{a'}{a}\delta'_C - \frac{3}{2}\left(\frac{a'}{a}\right)^2 (\Omega_C\delta_C + \Omega_B\delta_B) = 0\,,$$
$$\delta''_B + \frac{a'}{a}\delta'_B - \left[\frac{3}{2}\left(\frac{a'}{a}\right)^2 (\Omega_C\delta_C + \Omega_B\delta_B) - c_s^2 k^2\delta_B\right] = 0\,,$$

(b) For a small baryon density ($\Omega_B \ll \Omega_C$) and initially adiabatic perturbations, show that baryonic structures can only grow on physical wavelengths greater than,

$$\lambda_J \approx c_s\left(\frac{\pi}{G\bar{\rho}_C}\right)^{1/2}\,,$$

where $\bar{\rho}_C$ is the homogeneous cold dark matter density. Qualitatively describe the evolution of large-scale baryonic perturbations in the matter era ($t > t_{eq}$), given that $c_s \approx \mathcal{O}(1/\sqrt{3})$ prior to photon decoupling ($t < t_{dec}$) and $c_s \approx 10^{-5}(T/T_{dec})$ afterwards ($t > t_{dec}$). For $z_{dec} \approx 1000$ and $\Omega_B \approx 0.1\Omega_c$, roughly estimate the baryonic Jeans mass in solar masses just before decoupling and just after decoupling. [You may use $t_0 \approx 3\times10^{18}$s, $c = 3\times10^{10}\mathrm{cm\,s^{-1}}$, $M_{sun} \approx 10^{33}$g and $G = 10^{-7}\mathrm{cm^3g^{-1}s^{-2}}$.]

6 Consider the evolution equation for a scalar field ϕ in a flat ($k = 0$) FRW background,

$$\ddot{\phi} + 3H\dot{\phi} - \nabla^2\phi = -\frac{dV}{d\phi}\,, \qquad (*)$$

where $V(\phi)$ is the potential and the Hubble parameter $H = \dot{a}/a$.

(a) Perturb eqn $(*)$ about a homogeneous field $\phi(\mathbf{x}, t) = \phi(t) + \delta\phi(\mathbf{x}, t)$ to find the appropriate perturbation equation satisfied by $\delta\phi(\mathbf{x}, t)$; Fourier expand as $\delta\phi(\mathbf{x}, t) = \sum_{\mathbf{k}} \delta\phi_{\mathbf{k}}(t) \exp(i\mathbf{k}\cdot\mathbf{x})$ where $\mathbf{k}$ is the comoving wavevector ($k = |\mathbf{k}|$). (You may ignore any metric fluctuations.)

(b) Discuss the quantization of these fluctuations $\delta\phi_{\mathbf{k}}(t)$ in a large box of sidelength L expanding in annihilation and creation operators as $\delta\phi_{\mathbf{k}}(t) = w_{\mathbf{k}}(t)a_{\mathbf{k}} + w^*_{\mathbf{k}}(t)a^\dagger_{\mathbf{k}}$, which satisfy appropriate commutation relations. In a nearly de Sitter background ($a \propto \exp(Ht)$, $H \approx$const.), verify that the appropriate solution for the mode functions is given by

$$\omega_k(t) = L^{-3/2}\frac{H}{(2k^3)^{1/2}}\left(i + \frac{k}{aH}\right)\exp\left(\frac{ik}{aH}\right)\,. \qquad (\ddagger)$$

(c) In the small-scale limit ($k \gg aH$), demonstrate explicitly the reduction of ($\ddagger$) to the flat space result for a harmonic oscillator. On large scales after horizon exit ($k \ll aH$), show that the perturbations 'freeze' and explain why the variance $(\Delta\phi)^2|_k$ is scale-free.

7 Scalar metric perturbations in the synchronous gauge $h^{0\mu} = 0$, can be decomposed into two components h and h_S in Fourier space as

$$h_{ij} = \tfrac{1}{3}\delta_{ij}h + (\hat{k}_i\hat{k}_j - \tfrac{1}{3}\delta_{ij})h_S\,,$$

where the normalized wavevector $\hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}$. In the conformal Newtonian gauge, we can express the scalar metric perturbations in terms of two potentials Φ, Ψ as

$$ds^2 = a^2(\tau)\left[(1+2\Phi)d\tau^2 - (1-2\Psi)\delta_{ij}dx^i dx^j\right]\,.$$

(a) Consider gauge transformations, Fourier expanded as

$$\tilde{\tau} = \tau + \sum_k T_k(\tau)e^{i\mathbf{k}\cdot\mathbf{x}}\,, \qquad \tilde{\mathbf{x}} = \mathbf{x} + \sum_k L_k(\tau)i\hat{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}}\,,$$

to show that we can move from the synchronous gauge to the Newtonian gauge using

$$T_k = \frac{h_S'}{2k^2}\,, \qquad L_k = \frac{h_S}{2k}\,.$$

Give the potentials Φ and Ψ explicitly in terms of h and h_S and their derivatives.

(b) You are also given that photon density and velocities in the Newtonian gauge are related to their synchronous gauge counterparts by [*do not derive these*]

$$\delta_\gamma^N = \delta_\gamma + 2\frac{a'}{a}\frac{h_S'}{k^2}\,, \qquad \mathbf{v}_\gamma^N = \mathbf{v}_\gamma + i\mathbf{k}\,\frac{h_S'}{2k^2}\,,$$

and you may assume that there are no anisotropic stresses $\Phi = \Psi$. Use these results and those derived in (a) to show that scalar CMB temperature fluctuations in the direction $\hat{\mathbf{n}}$ in synchronous gauge,

$$\frac{\Delta T}{T}(\hat{\mathbf{n}}) = \tfrac{1}{4}\delta_\gamma + \mathbf{v}_\gamma\cdot\hat{\mathbf{n}} + \tfrac{1}{2}\int_{\tau_{\rm dec}}^{\tau_0} d\tau \sum_k e^{i\mathbf{k}\cdot\hat{\mathbf{n}}\tau}\left[\frac{1}{3}(h' - h_S') - (\hat{\mathbf{k}}\cdot\hat{\mathbf{n}})^2 h_S'\right]\,,$$

can be re-expressed in Newtonian gauge as

$$\frac{\Delta T}{T}(\hat{\mathbf{n}}) = \tfrac{1}{4}\delta_\gamma^N + \mathbf{v}_\gamma^N\cdot\hat{\mathbf{n}} + \Phi + 2\int_{\tau_{\rm dec}}^{\tau_0}\Phi' d\tau\,. \qquad (\dagger)$$

Briefly explain the physical significance of each of the terms in (†) and the angular scales on which they are important.

PART III COSMOLOGY — INFORMATION SHEET

You may make free use of the information on this sheet

The Friedmann-Robertson-Walker line element is

$$ds^2 = dt^2 - a^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right\}.$$

The Einstein and energy conservation equations can be written as:

$$\begin{aligned} &\text{Friedmann} && \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2}\,, \\ &\text{Raychaudhuri} && \frac{\ddot a}{a} = -\frac{4\pi G}{3}(\rho + 3P) + \frac{\Lambda}{3}\,, \\ &\text{Energy conservation} && \dot\rho + 3\frac{\dot a}{a}(\rho + P) = 0\,, \end{aligned}$$

with dots denoting derivatives with respect to cosmic time t.

Solutions to these equations in a flat FRW universe ($k = \Lambda = 0$) are:

$$\begin{aligned} &\text{Radiation}\quad P = \rho/3: && a \propto t^{1/2}\,, && \rho_{\rm crit} = \frac{3}{32\pi G t^2}\,, \\ &\text{Matter}\quad P \approx 0: && a \propto t^{2/3}\,, && \rho_{\rm crit} = \frac{1}{6\pi G t^2}\,, \end{aligned}$$

with the redshift of the radiation–matter transition at roughly $z_{\rm eq} \approx 10^4$.

For a relativistic particle species (mass m, chemical potential μ) with temperature $T \gg m, \mu$, the limiting energy, number and entropy equilibrium distributions yield

$$\begin{aligned} &\text{Bosons:} && \rho = \frac{\pi^2}{30} g T^4 && n = \frac{\zeta(3)}{\pi^2} g T^3\,, && s = \frac{2\pi^2}{45} g T^3\,, \\ &\text{Fermions:} && \rho = \frac{7}{8}\frac{\pi^2}{30} g T^4 && n = \frac{3}{4}\frac{\zeta(3)}{\pi^2} g T^3\,, && s = \frac{7}{8}\frac{2\pi^2}{45} g T^3\,, \end{aligned}$$

where g is the number of spin degrees of freedom of the particle and $\zeta(3) \approx 1.2$.

For a non-relativistic particle species $T \ll m$, the limiting particle number density is given by

$$n = g\left(\frac{mT}{2\pi}\right)^{3/2} \exp\left[-(m-\mu)/T\right].$$

We can relate the temperature and the Hubble parameter by

$$H = 1.66\mathcal{N}^{1/2}\frac{T^2}{m_{\rm pl}}\,,$$

where $\mathcal{N}$ is the number of effective massless degrees of freedom at a temperature T and the Planck mass $m_{\rm pl} = 1.2 \times 10^{19}$GeV.