UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 6 June 2002 9 to 12

PAPER 7

BANACH ALGEBRAS

*Attempt **THREE** questions*

*There are **six** questions in total*

The questions carry equal weight

All Banach algebras should be taken to be over the complex field.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let A be a Banach algebra with identity element 1. For an element $x \in A$, let

$$R_A(x) = \mathbb{C} \setminus \operatorname{Sp}_A x = \{\lambda \in \mathbb{C} : \lambda 1 - x \text{ is invertible in } A\}.$$

Let B be a closed subalgebra of A, containing 1, and let $x \in B$.

(i) Prove that $R_B(x)$ is a (relatively) open-and-closed subset of $R_A(x)$.

(ii) Deduce that, if U is a component of $R_A(x)$, then either U is a component of $R_B(x)$ or $U \subseteq \operatorname{Sp}_B x$.

(iii) Prove that $R_B(x)$ and $R_A(x)$ have the same unbounded component.

(iv) Let B be the closed subalgebra of A generated by 1 and x. Prove that every bounded component of $R_A(x)$ is a subset of $\operatorname{Sp}_B x$.

(v) Let x be invertible in A and suppose that $\operatorname{Sp}_A x$ does not separate 0 from ∞ (i.e. 0 belongs to the unbounded component of $R_A(x)$). Prove that there is a sequence (p_n) of complex polynomials such that $p_n(x) \to x^{-1}$ as $n \to \infty$.

2 Let A be a Banach algebra with identity, let $x \in A$ and let U be an open neighbourhood of $\operatorname{Sp} x$ in $\mathbb{C}$. Prove that there is a unique continuous, unital homomorphism $\Theta_x : \mathcal{O}(U) \to A$ such that $\Theta_x(Z) = x$ (where Z is the function $Z(\lambda) = \lambda$ $(\lambda \in U)$).
[*Any form of the Runge approximation theorem may be quoted without proof.*]

Let $g \in \mathcal{O}(U)$ and let $y = \Theta_x(g)$. Let h be holomorphic on open $V \supseteq g(U)$; explain why $V \supset \operatorname{Sp} y$, and prove that $\Theta_y(h) = \Theta_x(h \circ g)$ (where $\Theta_y : \mathcal{O}(V) \to A$ is the obvious functional calculus homomorphism).

Deduce that, if x is an invertible element of A and if $\operatorname{Sp} x$ does not separate 0 from ∞ (see Question 1 (v)), then $x = e^y$ for some element y of A.

Deduce that, if α is an invertible, $n \times n$ complex matrix, then $\alpha = e^\beta$ for some $n \times n$ matrix β.

3 Give an account of the elementary theory of C^*-algebras, leading to a proof of the Gelfand-Naimark theorem for commutative C^*-algebras.

Let A be a C^*-algebra and let h be a hermitian element of A. Prove that $\operatorname{Sp} h \subset \mathbb{R}^+$ if and only if $h = k^2$ for some hermitian element k of A.

Prove also that, if x is an arbitrary hermitian element of A, then there are hermitian elements $u, v \in A$ such that $x = u^2 - v^2$.

4 Let T be a bounded, normal operator on a Hilbert space H and let $B(\mathrm{Sp}\,T)$ be the algebra of all complex-valued, bounded Borel functions on $\mathrm{Sp}\,T$. Prove that there is a norm-decreasing, unital, $*$-homomorphism $\beta_T : B(\mathrm{Sp}\,T) \to \mathcal{B}(H)$ such that $\beta_T(Z) = T$ (where $Z(\lambda) = \lambda$ $(\lambda \in \mathrm{Sp}\,T)$).

[*All relevant results about compact Hausdorff spaces and C^*-algebras may be quoted without proof.*]

Prove that every bounded normal operator has a normal square root.

5 Write an account of the theory of irreducible representations of Banach algebras, up to an outline of B. E. Johnson's proof that every irreducible, normed representation of a Banach algebra is continuous.

6 (i) Let A be a Banach algebra and let p be a polynomial with coefficients from A. Prove that, for every $R > 1$,

$$r_A\big(p(1)\big)^2 \leqslant \sup_{|z|=R} r_A\big(p(z)\big) . \sup_{|z|=1/R} r_A\big(p(z)\big) .$$

(ii) Let A, B be Banach algebras, with B semisimple, and let $T : A \to B$ be a homomorphism with $T(A) = B$. Use the result of (i) to prove that T is continuous.