UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 3 June 2002 1.30 to 3.30

PAPER 69

PHASE TRANSITIONS AND COLLECTIVE PHENOMENA

Attempt ***THREE*** *questions*

There are ***four*** *questions in total*

The questions carry equal weight

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Explain the concept of *spontaneous symmetry breaking* and describe the circumstances under which low energy fluctuations are described by *Goldstone modes*?

The thermal fluctuations of an approximately flat membrane (i.e. one with no 'overhangs') embedded in d-dimensions can be described by its height $h(\mathbf{x})$ as a function of the remaining $d-1$ coordinates $\mathbf{x} = (x_1, \cdots, x_{d-1})$. If held under constant tension σ, the microscopic Hamiltonian is defined simply by $H = \sigma A$, where A denotes the total surface area

$$A[h(\mathbf{x})] = \int \mathrm{d}^{d-1}\mathbf{x} \left[1 + (\nabla h)^2\right]^{1/2}.$$

(a) At sufficiently low temperatures, fluctuations of the membrane involve only small, slowly varying field configurations $h(\mathbf{x})$. By expanding the Hamiltonian to *quadratic order* in h, show that the partition function can be expressed as a functional integral $\mathcal{Z} = \int Dh(\mathbf{x}) \exp\{-\beta H[h(\mathbf{x})]\}$ where, up to an irrelevant constant,

$$\beta H[h(\mathbf{x})] = \frac{\beta\sigma}{2} \int \mathrm{d}^{d-1}\mathbf{x}\ (\nabla h)^2.$$

(b) Applying a Fourier decomposition, show that the quadratic Hamiltonian is brought to diagonal form. Show that low-energy excitations (known as capillary waves) are described by Goldstone modes. Identify the continuous symmetry that is broken.

(c) Obtain an expression for the correlation function $\langle (h(\mathbf{x}) - h(0))^2 \rangle$ and comment on the physical implications of your result in dimensions $d = 2$, 3 and 4.

(d) Describe how the correlation function $\langle (h(\mathbf{x}) - h(0))^2 \rangle$ would differ if the membrane were bound by an additional quadratic potential,

$$V[h(\mathbf{x})] = \frac{t}{2} \int \mathrm{d}^{d-1}\mathbf{x}\ h^2(\mathbf{x})$$

2 Briefly outline the conceptual basis of the Renormalisation Group.

In the quadratic approximation, the Ginzburg-Landau Hamiltonian for the high temperature phase of a smectic takes the form

$$\beta H[m(\mathbf{x})] = \int dx_\parallel \int d^{d-1}\mathbf{x}_\perp \left[\frac{K}{2}(\nabla_\parallel m)^2 + \frac{L}{2}(\nabla_\perp^2 m)^2 + \frac{t}{2}m^2 - hm\right]$$

where $m(\mathbf{x})$ represents a one-component field depending on a d-dimensional set of coordinates $\mathbf{x} = (x_\parallel, \mathbf{x}_\perp)$, and the coefficients are constrained such that $K > 0$, $L > 0$ and $t > 0$.

(a) Express βH in terms of the Fourier coefficients $m(q_\parallel, \mathbf{q}_\perp)$.

(b) Construct a Renormalisation Group transformation for βH by rescaling the coordinates such that $q'_\parallel = bq_\parallel$, $\mathbf{q}'_\perp = c\mathbf{q}_\perp$, and the field $m' = m/z$.

(c) Choosing c and z such that $K' = K$ and $L' = L$, determine the scaling exponents y_t and y_h of the coefficents t and h at the resulting fixed point.

(d) Write down the relationship between the free energies $f(t, h)$ and $f(t', h')$ of the original and rescaled Hamiltonians. Hence write the unperturbed free energy in the homogeneous form

$$f(t, h) = t^{2-\alpha} g_f(h/t^\Delta),$$

and identify the exponents α and Δ.

3 Write detailed notes on **one** of the following:

(a) the Scaling Theory;

(b) the Kosterlitz-Thouless transition;

(c) Ginzburg-Landau Theory and the Ginzburg Criterion.

4 The one-dimensional lattice Ising ferromagnet is described by the microscopic Hamiltonian

$$\beta H = -\sum_{ij} J_{ij}\sigma_i\sigma_j - h\sum_i \sigma_i,$$

where the spins $\sigma_i = \pm 1$, h denotes the magnetic field, and the exchange interaction varies with separation between sites i and j as $J_{ij} = Je^{-\kappa|i-j|}$ with $\kappa \ll 1$.

(a) By employing an appropriate *Hubbard-Stratonovich transformation*, show that the partition function can be expressed in the form

$$\mathcal{Z} = C\int_{-\infty}^{\infty}\prod_k dm_k \exp\left[-\sum_{ij} m_i[J^{-1}]_{ij}m_j + \sum_i \ln(2\cosh(2m_i + h))\right]$$

where C represents some unspecified constant.

(b) For the long-ranged model defined above, show that

$$\mathcal{Z} = C\int_{-\infty}^{\infty}\prod_k dm_k \exp\left[-\sum_j\left(\frac{1}{2J\sinh\kappa}(m_j - m_{j+1})^2 + U(m_j)\right)\right],$$

where $U(m) = \tanh(\kappa/2)m^2/J - \ln[2\cosh(2m+h)]$.

(c) Taking the continuum limit, show that the classical partition function is isomorphic to the quantum transition amplitude of a particle in a double well potential. Drawing on this correspondence, comment on the existence of long-range order in the Ising system and describe qualitatively the effect of the magnetic field h. Describe qualitatively how this classical to quantum correspondence translates to the d-dimensional Ising system.