UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 5 June 2002 9 to 12

# PAPER 66

# THE STANDARD MODEL

*Attempt* ***THREE*** *questions*

*There are* ***four*** *questions in total*

*The questions carry equal weight*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** Let $\mathbf{A}_\mu = (A_{1\mu}, A_{2\mu}, A_{3\mu})$ and $B_\mu$ be the gauge fields and $g$, $g'$, the gauge couplings associated with the Standard Model gauge groups $SU(2)$ and $U(1)_Y$ respectively. The Higgs field $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ is in the $T = \frac{1}{2}$ fundamental representation of $SU(2)$ and the $Y = \frac{1}{2}$ representation of $U(1)_Y$, where i$Y$ is the generator of that group.

Write down the corresponding representations of the known leptons (assuming absence of right-handed neutrinos). Describe the construction of gauge-invariant kinetic terms for leptons and the Higgs field, and interactions of leptons with the Higgs field, in the classical Lagrangian.

If the gauge symmetry is spontaneously broken by the choice of vacuum expectation value

$$\langle\phi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} ,$$

show that by suitable gauge transformation one may write

$$\phi(x) = \frac{v + \rho(x)}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} ,$$

for real scalar field $\rho$. Hence show from your Lagrangian in this gauge that diagonal mass terms arise for leptons and the fields $W_\mu = (A_{1\mu} - \mathrm{i}A_{2\mu})/\sqrt{2}$ and $Z_\mu = \cos\theta_W A_{3\mu} - \sin\theta_W B_\mu$, where $\tan\theta_W = g'/g$.

[The Pauli matrices are

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \ \tau_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} , \ \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \ ]$$

**2** The part of the Electro-Weak Lagrangian density that couples the $W_\mu(x)$ gauge field, of mass $M_W$, to the electron field $e(x)$ and its neutrino field $\nu_e(x)$ is

$$\mathcal{L}_W = \frac{g}{2\sqrt{2}}\left(W_\mu \overline{\nu}_e \gamma^\mu (1-\gamma_5)e \ + \ \text{h.c.}\right) .$$

Neglecting fermion masses, use $\mathcal{L}_W$ to calculate to lowest order in $g$ the unpolarized decay rate $\Gamma(W^- \to e\overline{\nu}_e) = g^2 M_W/48\pi$. Calculate also the result if the polarization vector is restricted to $\epsilon(p,\lambda) = (0,0,0,1)$ for a $W^-$ at rest.

[ The following may be assumed without proof:

the differential decay rate into final state particles of momentum $k_j = (k_j^0, \mathbf{k}_j)$ is, for amplitude $\mathcal{M}$,

$$d\Gamma = \frac{1}{2M_W}\int \prod_j \left[\frac{d^3\mathbf{k}_j}{(2\pi)^3 2k_j^0}\right](2\pi)^4\delta^4\Big(p - \sum_j k_j\Big)|\mathcal{M}|^2 \ ;$$

$$\begin{aligned}\text{Tr}\{\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\} &= -4\mathrm{i}\epsilon^{\mu\nu\rho\sigma} \ ,\\ \text{Tr}\{\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\} &= 4\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right) \ ;\end{aligned}$$

for $W^-$ of momentum $p$, polarization $\lambda$,

$$\begin{aligned}\langle W^-(p,\lambda)|W_\mu(0)|0\rangle &= \epsilon^*_\mu(p,\lambda) \ ,\\ \sum_\lambda \epsilon_\mu(p,\lambda)\epsilon^*_\nu(p,\lambda) &= -g_{\mu\nu} + \frac{p_\mu p_\nu}{M_W^2} \ . \end{aligned}$$ ]

**3** The quark field $q(x)$ has components $q_{\alpha i f}(x)$, with spinor index $\alpha \in \{1,2,3,4\}$, colour $SU(3)$ index $i \in \{1,2,3\}$, and flavour index $f \in \{1,2\}$ (assume only $u$ and $d$ quarks). In component form, write down the gauge-invariant kinetic term for massless quarks in the QCD Lagrangian, its gauge and global continuous symmetry transformations, and the canonical quantum anti-commutation relations for quark fields $q_{\alpha i f}(x)$.

If $\overline{q} \equiv q^\dagger\gamma^0$, where $\dagger$ acts on all indices, the axial symmetry generators may be written

$$Q_{5,a} = \frac{1}{2}\int d^3\mathbf{x}\ \overline{q}(x)\gamma^0\gamma_5\tau_a q(x) \quad ; \quad a \in \{1,2,3\} \ .$$

Defining fields

$$S(x) = \overline{q}(x)q(x) \quad \text{and} \quad P_a(x) = \mathrm{i}\overline{q}(x)\gamma_5\tau_a q(x) \ ,$$

prove the equal-time commutation relations

$$\begin{aligned}[Q_{5,a}, S(x)] &= \mathrm{i}P_a(x)\\ [Q_{5,a}, P_b(x)] &= -\mathrm{i}\delta_{ab}S(x) \ .\end{aligned}$$

Use these to discuss whether this axial symmetry is spontaneously broken in nature. State the effect of small quark masses $m_u = m_d \neq 0$ on all your results in this question.

**4 (a)** Explain why the renormalised coupling $\lambda(\mu)$ in a renormalisable field theory, with no dimensionful parameters in the classical Lagrangian, must be defined via an arbitrary dimensionful scale $\mu$ at the quantum level.

**(b)** Suppose $\mu$ is such that a perturbative expansion in small $\lambda(\mu)$ is valid for the $\beta$-function

$$\beta(\lambda) \equiv \mu \frac{d\lambda}{d\mu} = -\frac{\beta_0 \lambda^3}{(4\pi)^2} - \frac{\beta_1 \lambda^5}{(4\pi)^4} + \mathrm{O}(\lambda^7) \ .$$

Defining $\alpha_\lambda \equiv \lambda^2/4\pi$, show that the solution is

$$\frac{4\pi}{\alpha_\lambda} = \beta_0 \ln \frac{\mu^2}{\Lambda^2} + \frac{\beta_1}{\beta_0} \ln \ln \frac{\mu^2}{\Lambda^2} + \mathrm{O}\left( \left( \ln \frac{\mu^2}{\Lambda^2} \right)^{-1} \right)$$

for suitable choice of $\Lambda$.

**(c)** In the $SU(5)$ unified gauge theory with coupling $g_5$, before spontaneous symmetry breaking to $SU(3) \times SU(2) \times U(1)_Y$, with gauge couplings $g_s$, $g$, $g'$ respectively, there is a relation $g_5 = g_s = g = -g'\sqrt{5/3}$. Using this as a boundary condition at some scale $\mu = M_X$ and assuming that $g_s$, $g$, $-g'\sqrt{5/3}$, run with the leading order perturbative $\beta$-function of their respective gauge groups for $\mu < M_X$, show that this leads to a prediction for $\sin^2\theta_W$ and $M_X$ in terms of $\alpha_s \equiv g_s^2/4\pi$ and $\alpha \equiv e^2/4\pi$.

[You may assume: $\beta_0 = (11N - 2n_f)/3$ for $SU(N)$, $\beta_0 = -2n_f/3$ for $U(1)$, where $n_f$ is number of quark flavours; $g' \equiv g \tan\theta_W$, $e \equiv g \sin\theta_W$.]