UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 31 May 2002 1.30 to 4.30

PAPER 65

ADVANCED QUANTUM FIELD THEORY

*Attempt **THREE** questions*

*There are **four** questions in total*

The questions carry equal weight

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A scalar field ϕ in d-dimensional spacetime has lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda_3}{3!}\phi^3 - \frac{\lambda_4}{4!}\phi^4 - \frac{\lambda_6}{6!}\phi^6 \ . \qquad (*)$$

State the Feynman rules for this theory, explaining how they are used to calculate momentum space correlation functions. [No derivations are required.]

What is meant by the number of loops L in a Feynman diagram? Show that a connected diagram is proportional to $\hbar^{L-1}$ for fixed m and λ_i. Show also that if V is the total number of vertices in a connected diagram contributing to the n-point function in the theory $(*)$, then

$$\frac{V}{2} \leqslant L - 1 + \frac{n}{2} \leqslant 2V \ .$$

Explain what is meant by a one-particle-irreducible Feynman diagram and illustrate your answer by drawing two connected diagrams contributing to the four-point function in $(*)$, one of which is one-particle-irreducible and the other of which is not. Why is this concept useful in analysing the divergences and renormalization of a quantum field theory?

Consider in turn the cases $d = 3, 4, 6$ and determine which of the interactions in $(*)$ are allowed if the theory is to be renormalizable. [State clearly any general result you wish to use.]

2 (a) Evaluate

$$\int du_1 \ldots \int du_n \exp(-\tfrac{1}{2}u_i A_{ij} u_j + b_i u_i)$$

where u_i and b_i are real, n-component vectors and A_{ij} is a real $(n{\times}n)$ matrix which is symmetric and positive-definite. [You may quote the result for $n = 1$ and $b_i = 0$.]

(b) Describe how linear operators, including the notions of the identity and inverse operators, generalize from finite-dimensional vectors u_i to real fields $\phi(x)$. Define what is meant by functional differentiation.

(c) Assuming a generalization of your answer to part (a), calculate the generating functional $Z_0[J]$ for a free scalar field with the lagrangian $\mathcal{L}_0(\phi) = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2$, deriving the form of the free propagator in momentum space.

(d) Explain, by first quoting an appropriate finite-dimensional result, how to express $Z[J]$, the generating functional for an interacting theory with the lagrangian $\mathcal{L}(\phi) = \mathcal{L}_0(\phi) + \mathcal{L}_I(\phi)$, in terms of $Z_0[J]$.

3 Define the β-function for a quantum field theory with a dimensionless, renormalized coupling $\lambda(\mu)$, where μ is the renormalization scale. Discuss the behaviour of $\lambda(\mu)$ if $\beta = b\lambda^2 + O(\lambda^3)$, distinguishing the cases in which the constant b is positive or negative, and explaining the term asymptotic freedom.

A real scalar field ϕ in four dimensions has mass m and interaction lagrangian $-\frac{\lambda}{4!}\phi^4$. The one-loop contribution to the amputated four-point function in renormalized perturbation theory is a sum of three integrals, each of the form

$$\frac{i\lambda^2}{2(2\pi)^4} \int \frac{d^4k}{(\,(k+p)^2 + m^2\,)(k^2 + m^2)}\ ,$$

where p is a certain combination of external momenta, and all momenta have been Wick-rotated to Euclidean space. Use dimensional regularization and minimal subtraction to evaluate the divergent part of the integral above and to obtain the total one-loop counterterm. Calculate the β-function for this theory at one loop.

[$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \sim \frac{1}{z} + O(1)$ for suitable z.]

4 Starting from a Lie algebra with anti-hermitian generators T_a obeying $[T_a, T_b] = f_{abc} T_c$ and $\mathrm{Tr}(T_a T_b) = -\delta_{ab}$, define gauge transformations with parameters ω_a acting on gauge fields A^a_μ and obtain an expression for infinitesimal gauge transformations by expanding your definition to first order in ω_a.

Consider the functional integral

$$Z = \int \mathcal{D}A^a_\mu \, e^{iS[A^a_\mu]} \, \delta[\mathcal{F}_a(A^a_\mu)] \, \Delta(A^a_\mu)\ .$$

where $S[A^a_\mu]$ is a gauge-invariant action. Explain the origin of the factors $\delta[\mathcal{F}_a]$ and $\Delta(A^a_\mu)$ and derive a formula for the latter as a functional determinant.

Indicate, very briefly, how Z can be re-written so that both $\delta[\mathcal{F}_a]$ and $\Delta(A^a_\mu)$ are replaced by modifications to the original action $S[A^a_\mu]$.