UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 4 June 2002 1.30 to 4.30

PAPER 63

ELEMENTARY PARTICLE PHYSICS

*Attempt **THREE** questions*

*There are **four** questions in total*

The questions carry equal weight

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Isospin operators $\hat{I}_\pm$, $\hat{I}_3$ satisfy the algebra $[\hat{I}_3, \hat{I}_\pm] = \pm\hat{I}_\pm$, $[\hat{I}_+, \hat{I}_-] = 2\hat{I}_3$. The isospin states $|II_3\rangle$ satisfy

$$\hat{I}_3|II_3\rangle = I_3|II_3\rangle\,, \qquad \hat{I}_\pm|II_3\rangle = \big((I \mp I_3)(I \pm I_3 + 1)\big)^{\frac{1}{2}}|II_3\rangle\,.$$

For two $I = 1$ states $|1m\rangle_1$ and $|1m'\rangle_2$ construct a tensor product state with total isospin $I = 0$. Without detailed calculation explain why the $I = 0, 2$ states are symmetric and the $I = 1$ state is antisymmetric.

For two pions explain why if the relative orbital angular momentum ℓ is odd then they are in a total isospin $I = 1$ state.

The K^0 is a spinless particle with $I = \frac{1}{2}$, $I_3 = -\frac{1}{2}$. It decays into two pions with a matrix element

$$\langle \pi_1, \pi_2|H_I|K^0\rangle\,,$$

where H_I is an interaction such that $[\hat{I}_3, H_I] = \frac{1}{2}H_I$, $[\hat{I}_+, H_I] = 0$. Explain why the two pions must be in a $I = 0$ state. What is the value expected for $\Gamma_{K^0\to\pi^+\pi^-}/\Gamma_{K^0\to\pi^0\pi^0}$?

2 $SU(3)$ tensors $T^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_l}$, where $\alpha_i, \beta_j \in \{1, 2, 3\}$, are defined by the transformation rules

$$T^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_l} \to \prod_{i=1}^{k} A^{\alpha_i}{}_{\gamma_i}\, T^{\gamma_1\ldots\gamma_k}_{\delta_1\ldots\delta_l} \prod_{j=1}^{l} (A^{-1})^{\delta_j}{}_{\beta_j}\,,$$

where $A^\alpha{}_\gamma$ are elements of a 3×3 unitary matrix with determinant 1.

Show that $\delta^\alpha{}_\beta$, $\epsilon_{\alpha\beta\gamma}$, $\epsilon^{\alpha\beta\gamma}$ are invariant tensors. Show how the tensors $T^{\alpha\beta\gamma}$ and T^α_β may be decomposed in terms of tensors which transform irreducibly.

Describe how the decomposition of the tensor $T^{\alpha\beta\gamma}$ is relevant to the description of observed baryons as constructed from u, d, s quarks.

3 A simple Lie algebra has elements $\{H_i,\, E_{\boldsymbol{\alpha}}\}$ satisfying

$$[H_i, H_j] = 0\,, \quad i,j = 1,\dots r\,, \qquad [H_i, E_{\boldsymbol{\alpha}}] = \alpha_i E_{\boldsymbol{\alpha}}\,,$$

where $\{H_i\}$ form a maximal set of commuting elements, so that r is the rank, and $\{\boldsymbol{\alpha}\}$ are the roots. Assume also that

$${H_i}^\dagger = H_i\,, \qquad {E_{\boldsymbol{\alpha}}}^\dagger = E_{-\boldsymbol{\alpha}}\,.$$

Show, using the Jacobi identity, that

$$[E_{\boldsymbol{\alpha}}, E_{\boldsymbol{\beta}}] = \begin{cases} \tilde{\alpha}_i H_i\,, & \text{if } \boldsymbol{\beta} = -\boldsymbol{\alpha}\,, \\ e_{\boldsymbol{\alpha},\boldsymbol{\beta}}\, E_{\boldsymbol{\alpha}+\boldsymbol{\beta}}\,, & \text{if } \boldsymbol{\alpha}+\boldsymbol{\beta} \text{ is a root} \end{cases},$$

for some $\tilde{\alpha}_i$ and $e_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. Assume $\tilde{\alpha}_i = 2\alpha_i/\boldsymbol{\alpha}^2$. Let $H_{\boldsymbol{\alpha}} = 2\alpha_i H_i/\boldsymbol{\alpha}^2$. Show that $E_{\pm\boldsymbol{\alpha}}$, $H_{\boldsymbol{\alpha}}$ form an $SU(2)$ algebra and, using standard results, that $H_{\boldsymbol{\alpha}}$ has integer eigenvalues.

Let X_λ be elements of the Lie algebra such that $[H_{\boldsymbol{\alpha}}, X_\lambda] = \lambda X_\lambda$. Why must $[E_{\boldsymbol{\alpha}}, X_\lambda] \propto X_{\lambda+2}$ or $[E_{\boldsymbol{\alpha}}, X_\lambda] = 0$? Assume then that $X_\lambda \in \{X_{-n}, X_{-n+2}, \dots, X_{n-2}, X_n\}$ for some integer $n \geq 0$. Hence show that, if $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are roots, then $2\boldsymbol{\alpha}\cdot\boldsymbol{\beta}/\boldsymbol{\alpha}^2$ is an integer and also $\boldsymbol{\beta} - (2\boldsymbol{\alpha}\cdot\boldsymbol{\beta}/\boldsymbol{\alpha}^2)\boldsymbol{\alpha}$ is a root.

What are the possible angles between roots $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and also the possible values for $\boldsymbol{\alpha}^2/\boldsymbol{\beta}^2$? [You may assume that if $\boldsymbol{\alpha}$ is a root then $2\boldsymbol{\alpha}$ is not].

For $r = 2$ draw the possible root diagrams. What are simple roots? Identify possible simple roots on each root diagram. Define the Cartan matrix in terms of the simple roots and work it out in each case.

4 For a Lie algebra with a Lie bracket $[T_a, T_b] = c^c{}_{ab} T_c$, $a, b, c = 1, \dots, D$ explain what is the adjoint representation? Show that there are corresponding $D \times D$ matrices T_a^{ad} which obey the Lie algebra.

Let $A_\mu^a(x)$ be a vector field (μ is a Lorentz index) and define

$$F_{\mu\nu}^a(x) = \partial_\mu A_\nu^a(x) - \partial_\nu A_\mu^a(x) + c^a{}_{bc} A_\mu^b(x) A_\nu^c(x)\,.$$

If $\delta A_\mu^a(x) = \partial_\mu \lambda^a(x) + c^a{}_{bc} A_\mu^b(x) \lambda^c(x)$ work out $\delta F_{\mu\nu}^a(x)$.

Define $\kappa_{ab} = \mathrm{tr}\big(T_a^{\mathrm{ad}}\, T_b^{\mathrm{ad}}\big)$. Show that

$$c^d{}_{ca}\, \kappa_{db} + c^d{}_{cb}\, \kappa_{ad} = 0\,,$$

and hence that $\delta\big(F^{a\mu\nu} F_{\mu\nu}^b\, \kappa_{ab}\big) = 0$.