UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 4 June 2002 1.30 to 4.30

PAPER 61

APPROXIMATION THEORY

*Attempt **FIVE** questions*

*There are **seven** questions in total*

The questions carry equal weight

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Korovkin's theorem states:

If (U_n) is a sequence of positive linear operators on $C[0,1]$ such that

$$U_n(p_k) \to p_k \quad \text{on} \quad p_k(x) = x^k, \quad k = 0, 1, 2,$$

then

$$U_n(f) \to f \quad \forall\, f \in C[0,1].$$

The main stage of its proof is the following statement:

For any $f \in C[0,1]$ and for any $\epsilon > 0$ there exists a constant $\gamma = \gamma(f, \epsilon)$ such that, with

$$q_t^{\pm}(x) := f(t) \pm [\epsilon + \gamma(x-t)^2],$$

we have the inequalities

$$q_t^-(x) < f(x) < q_t^+(x), \quad \forall x, t \in [0,1].$$

a) Starting from this stage complete the proof of Korovkin theorem.

b) Prove that the only positive linear operator U on $C[0,1]$ such that

$$U(p) = p \quad \text{for all quadratic functions} \quad p(x) = ax^2 + bx + c,$$

is the identity operator I such that $I(f) = f$ for all $f \in C[0,1]$.

2 Let T_n be the Chebyshev polynomial of degree n, let $\Delta^* := (t_i^*) := (\cos \frac{\pi i}{n})_{i=0}^n$ be the sequence of its equioscillation points, and let $\|\cdot\| := \|\cdot\|_{C[-1,1]}$.

According to the Markov-Duffin-Schaeffer theorem, if p_n is a polynomial of degree n which satisfies

$$|p_n(t_i^*)| \le 1, \qquad t_i^* \in \Delta^*,$$

then

$$\|p_n^{(k)}\| \le |T_n^{(k)}(1)|, \qquad k = 1, \ldots, n\,.$$

Prove that Δ^* is the only sequence with this property, i.e., for any other sequence $\Delta = (t_i)_{i=0}^n \subset [-1,1]$ with distinct t_i which differs from Δ^* at least in one point, there exists a polynomial q_n of degree n such that

$$|q_n(t_i)| \le 1, \qquad t_i \in \Delta\,,$$

and

$$\|q_n^{(k)}\| > |T_n^{(k)}(1)|, \qquad k = 1, \ldots, n\,.$$

[Hint: Use the Lagrange interpolation formula, certain sign patterns for $q_n(t_i)$, and the inequality $\|q_n^{(k)}\| \ge |q_n^{(k)}(1)|$.]

3 Let $E_n(f)$ be the value of the best approximation of a 2π-periodic f by trigonometric polynomials of degree n, and let $\omega(f,\delta)$ be the modulus of continuity of f.

Formulate the inverse theorem for trigonometric approximation and show that

$$E_n(f) = \mathcal{O}(n^\alpha) \quad \text{implies} \quad \omega(f,\delta) = \mathcal{O}(\delta^\alpha), \qquad 0 < \alpha < 1.$$

You should pay attention to the values $\delta > 1$ and $\frac{1}{n+1} < \delta < \frac{1}{n}$.

Find the order of $\omega(g,\delta)$ for the Weierstrass function

$$g(x) := \textstyle\sum_{k=0}^{\infty} \frac{1}{2^k} \cos 3^k x$$

using the fact that, for $3^m \le n < 3^{m+1}$, the polynomial of best approximation of degree n to g is the partial sum $t_n(x) = \sum_{k=0}^{m} \frac{1}{2^k} \cos 3^k x$.

4 Let $(N_{i,k})_{i=0}^n$ be the sequence of B-splines of order k on the uniform knot sequence $\Delta = (t_i)_{i=0}^{n+k} = (0, 1, \ldots, n+k)$.

(a) Write down the recurrence relation between $N_{i,k}$ and $N_{j,k-1}$.

(b) Use it to determine the values of $N_{0,k}(x)$ at its knots for $k = 3, 4, 5, 6$. Arrange results in the triangular array

$$\begin{array}{lccccccc}
k=2 & & & 0 & 1 & 0 & & \\
k=3 & & 0 & * & & * & 0 & \\
\vdots & & & & \vdots & & & \\
k=6 & 0 & * & * & * & * & * & 0
\end{array}$$

(c) Consider the interpolation problem of finding $s = \sum_{i=0}^n a_i N_{i,k}$ such that

$$s(x_i) = f(x_i), \qquad x_i = \frac{t_i + t_{i+k}}{2}, \qquad i = 0, \ldots, n,$$

Let $A_x a = f\big|_x$ be the linear system for determining (a_i). For $k = 6$ write down the matrix A_x and prove that the ℓ_∞-norm of its inverse satisfies

$$\|A_x^{-1}\|_\infty \le 10.$$

[*You may use any appropriate theorems on the inverse of certain matrices if correctly stated.*]

5 State the Chebyshev alternation theorem on the element of best approximation to a function $f \in C[0,1]$ from $\mathcal{P}_n$, the space of all algebraic polynomials of degree n.

Let

$$E_n(f) := \inf_{p_n \in \mathcal{P}_n} \|f - p_n\|_{C[0,1]}.$$

It is clear that, for any $f \in C[0,1]$, we have the inequality

$$E_n(f) \geq E_{n+1}(f).$$

Prove that, if $f \in C^{n+1}[0,1]$ and $f^{(n+1)} > 0$ on $[0,1]$, then

$$E_n(f) > E_{n+1}(f),$$

i.e., for such f the equality sign is excluded.

6 Given a knot sequence $\Delta = (t_i)_{i=1}^{n+k}$, let ω_i and $\ell_i(\cdot, t)$ be polynomials in $\mathcal{P}_{k-1}$ defined by

1) $\omega_i(x) := (x - t_{i+1}) \cdots (x - t_{i+k-1})$,
2) $\ell_i(\cdot, t)$ interpolates $(\cdot - t)_+^{k-1}$ on $x = t_i, ..., t_{i+k-1}$,

and let

$$N_i := (t_{i+k} - t_i)[t_i, \ldots, t_{i+k}](\cdot - t)_+^{k-1}$$

be the B-spline of order k with the knots $t_i, \ldots, t_{i+k}$.

Prove Lee's formula

$$\omega_i(x) N_i(t) = \ell_{i+1}(x, t) - \ell_i(x, t), \quad \forall x, t \in \mathbb{R}$$

and derive from it the Marsden identity:

$$(x - t)^{k-1} = \sum_{i=1}^{n} \omega_i(x) N_i(t), \quad t_k < t < t_{n+1}, \quad \forall x \in \mathbb{R}.$$

7 (1) Let $\mathbb{X}$ be an innner product space with the scalar product $(\cdot,\cdot)$ and the norm $\|x\| := (x,x)^{1/2}$, and let $\mathcal{U}_n$ be an n-dimensional subspace.

(a) Prove that $u^* \in \mathcal{U}_n$ is the best approximation to $x \in \mathbb{X}$ from $\mathcal{U}_n$ if and only if

$$(x - u^*, v) = 0 \quad \forall v \in \mathcal{U}_n.$$

(b) Let $(u_j)_{j=1}^n$ be a basis for $\mathcal{U}_n$. Derive the normal equations for determining the coefficients of expansion $u^* = \sum_j a_j u_j$.

(2) Given $f \in C[0,1]$ and a basis (N_j) of the L_∞-normalized B-splines, let

$$P_{\mathcal{S}}(f) := s^* = \sum_{j=1}^n a_j N_j$$

be the best spline approximation to f from $\mathcal{S} := \mathrm{span}\,(N_j)$ with respect to the L_2-norm, and $P_{\mathcal{S}}$ is also well defined as an operator from $C[0,1]$ onto $C[0,1]$.

Show that the max-norm of $P_{\mathcal{S}}$ satisfies the inequality

$$\|P_{\mathcal{S}}\|_\infty \le \|G^{-1}\|_{\ell_\infty}$$

where G is an appropriate Gram matrix.