UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 4 June 2002 9 to 12

PAPER 60

NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS

*Attempt **THREE** questions from Section A and **ONE** question from Section B*

*There are **five** questions in Section A and **two** in Section B*

Each question in Section A carries half the weight of each question in Section B

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

Section A

1 The Laplace operator ∇^2 is discretised in the unit square by the following two finite-difference methods,

$$\text{(i)}\quad \frac{1}{(\Delta x)^2}\begin{array}{ccc} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{array} \qquad \text{and (ii)}\quad \frac{1}{(\Delta x)^2}\begin{array}{ccc} \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ \frac{2}{3} & -\frac{10}{3} & \frac{2}{3} \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{array}$$

Here $\Delta x = 1/(m+1)$, while m is the number of internal grid points in each direction.

a. To each method there corresponds an $m^2 \times m^2$ matrix A, say, of finite differences. Evaluate explicitly the eigenvalues of A for both methods.

b. The eigenvalues of A can be used to approximate the eigenvalues of ∇^2 in the unit square, namely $-(k^2+l^2)\pi^2$, $k, l = 1, 2, \ldots$. How good is the approximation? Which method produces better estimate?

2 Prove that the highest order of a convergent s-step method for ODEs is $2[(s+2)/2]$. [You may use the Dahlquist equivalence theorem, as long as you state it precisely.]

3 (a) Determine the conditions on $\mathbf{c}$, $\mathbf{b}$ and A so that the Runge–Kutta method with the Butcher tableau

$$\begin{array}{c|c} \mathbf{c} & A \\ \hline & \mathbf{b}^\top \end{array}$$

is of order $p \geqslant 2$.

(b) Given the two-stage Runge–Kutta method with the Butcher tableau

$$\begin{array}{c|cc} c_1 & \frac{1}{2}\frac{c_1(-c_1+2c_2)}{c_2-c_1} & -\frac{1}{2}\frac{c_1^2}{c_2-c_1} \\ c_2 & -\frac{1}{2}\frac{c_2^2}{c_2-c_1} & \frac{1}{2}\frac{c_2(-2c_1+c_2)}{c_2-c_1} \\ \hline & \frac{c_2-\frac{1}{2}}{c_2-c_1} & \frac{\frac{1}{2}-c_1}{c_2-c_1} \end{array}$$

where $c_1, c_2 \in [0,1]$, $c_1 \neq c_2$, verify that it is always of order $p \geqslant 2$ and identify all values of c_1, c_2 such that the method is A-stable.

4 (a) Determine the order p of the *Strang splitting*

$$E(t; A, B) := \mathrm{e}^{\frac{1}{2}tA}\mathrm{e}^{tB}\mathrm{e}^{\frac{1}{2}tA} = \mathrm{e}^{t(A+B)} + O(t^{p+1}).$$

(b) We solve the diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \qquad t \geqslant 0, \quad x, y \in [0,1],$$

with given initial conditions at $t = 0$ and zero boundary conditions on the boundary of $[0,1]^2$ by the method

$$U^{n+1} = E(\mu; A, B)U^n,$$

where A and B are finite-difference matrices approximating the x and y derivatives, respectively, and μ is the Courant number. Prove that, for a natural choice of the matrices A and B, the method is stable.

5 (a) The finite-difference method

$$u_m^{n+1} = (1-2\mu)(u_m^n - u_{m+1}^n) + u_{m+1}^{n-1}, \qquad \mu = \frac{\Delta t}{\Delta x},$$

is used to solve numerically the advection equation $u_t = u_x$, with Dirichlet initial conditions given at $t = 0$ for all $-\infty < x < \infty$. Find the order for general μ.

(b) Determine the range of μ that gives stability.

Section B

6 Write an essay on Gaussian elimination of sparse linear algebraic systems and its relationship with graphs of matrices.

7 Write an essay on stability analysis of finite-difference methods for partial differential equations of evolution. Explain the importance of stability and survey briefly, with simple examples, the main techniques of stability analysis.