UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 30 May 2002 1.30 to 4.30

PAPER 6

INTRODUCTION TO FUNCTIONAL ANALYSIS

Attempt ***THREE*** *questions*

There are ***four*** *questions in total*

The questions carry equal weight

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Prove the Baire category theorem and use it to prove the Banach-Steinhaus theorem.

Use the Banach-Steinhaus theorem to show that there exists a continuous function whose Fourier series diverges at a given point.

2 Prove a version of the Hahn-Banach theorem for real vector spaces and use it to show that generalised limits exist.

3 Consider a commutative Banach algebra B with a unit. Develop the thory of the resolvent $R(x)$ of an element $x \in B$ up to and including the proof of the formula

$$\rho(x) = \sup\{\lambda : \lambda \notin R(x)\}$$

and prove the formula.

[You may assume results from the theory of vector valued integration but not from the theory of Banach algebra valued analytic functions.]

4 (i) Identify the maximal ideal spaces of the Weiner algebra $A(\mathbb{T})$ (the algebra of absolutely convergent Fourier series), of $A^+(\mathbb{T})$ (the algebra of absolutely convergent Taylor series) and of $C(\mathbb{T})$ (the algebra of continuous functions) on the circle $\mathbb{T}$.

[You may assume general theorems on Banach algebras if these are clearly stated but not results which reduce an identification to a triviality.]

(ii) State, but do not prove, a geometric version of the Hahn-Banach theorem and use it to prove the Krein-Milman theorem. You may restrict yourself to real normed vector spaces.