UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 6 June 2002 9 to 11

PAPER 57

POPULATION DYNAMICS

*Attempt **THREE** questions*

*There are **six** questions in total*

The questions carry equal weight

Additional credit is given for substantially complete answers

Wherever possible state any biological assumptions and give a biological interpretation to your results.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Ecology

Write down the standard Lotka-Volterra predator-prey model and describe its dynamics. Given that the predator and prey undergo small oscillations about the fixed point, discuss the stability of harvesting a fixed number of prey annually. Consider the dynamics if harvesting occurs when either (a) the prey reach a fixed abundance or (b) the predators reach a fixed abundance. Comment briefly on the implications of these models for conservation.

2 Ecology

A plant species (resource) and a herbivore (plant eater) can be modelled by the Lotka-Volterra equation. If a predator is now added to the system, then the herbivore will suffer an additional death rate from predation. Find the dynamics of this simple 3 species system.

(a) Show that a 4 species system (resource, herbivore, predator, top-predator) has a fixed point, and comment on how this is likely to generalise to a chain of N species.

(b) For the 4 species model, calculate the long-term average abundance of each species and show that all 4 species can co-exist. Is this result likely to hold for a general N species food chain?

(c) For a more complex food web, with Lotka-Volterra type interactions, are all species likely to persist?

3 Epidemiology

The recent Foot and Mouth outbreak in the UK can be modelled by assuming that each farm is either Susceptible, Infected or Recovered, forming a Simple Epidemic (no births or deaths). Let P_n be the proportion of farms with n livestock. For each of the following 3 assumptions, construct the model, find R_0 and sketch the dynamics of the epidemic in comparison to the homogeneous SIR model. You should also comment on how the average number of livestock per infected and susceptible farm changes over time.

(a) The susceptibility of a farm is proportional to n (but the infectivity is independent).

(b) The infectivity of a farm is proportional to n (but the susceptibility is independent).

(c) Both the susceptibility and the infectivity of a farm is proportional to n.

(The homogeneous model has the same transmission and susceptibility characteristics as the full model, but assumes that all farms have the average number of livestock.)

4 Epidemiology

Define R_0 in words and state the relationship between R_0, the invasibility of a population and the asymptotic level of susceptibles.

Chlamydia and Gonorrhoea are both STD (sexually transmitted diseases) and can be modelled using the SIS framework. New evidence shows that being infected by any one disease increases the risk (rate) of being infected by the other. Write down a suitable model for these two STDs and find the conditions for an allee effect to operate (the diseases cannot invade at low levels or in isolation). (For simplicity you may assume both diseases have the same infectious period and that recovery from either disease is unaffected by the presence of the other disease.)

5 Evolution

Two species of grass compete for habitat. Species A is a good coloniser of empty habitat (if A lands on an empty habitat the probability that it successfully establishes is higher than the corresponding probability for B), whereas species B is a good competitor (if B lands on habitat already occupied by A the chance that it replaces A is higher than the corresponding probability for A replacing B). If habitats are occasionally emptied by fire, construct a Levins metapopulation model for the three types of habitat (empty, A and B) and find the criterion for coexistence.

If colonisation and competition are linked (ie they are monotonic functions of a single evolutionary parameter) find the selective pressure and the conditions for a species to be a local or global ESS.

6 Evolution (Genetics)

Two cousins marry each other and produce children.

(a) What is the probability that one of the children has at least one homozygote chromosome?

(b) If normal siblings have a relatedness of $\frac{1}{2}$, what is the relatedness of the cousins' children? (Relatedness is defined as the probability that two individuals share a given chromosome.) How does this value change if you incorporate the fact that the cousins must be of opposite sex? [Note: the human genome has 23 chromosomes]

(c) In a given population of constant size N, let r_G be the average relatedness of any two individuals in generation G. In each generation, each male-female pair produces exactly two offspring before dying. Assuming that initially $r_0 = 0$, find the behaviour of r_G. How is this affected if siblings are prevented from breeding?