UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 7 June 2002 1.30 to 3.30

PAPER 55

SOLIDIFICATION OF FLUIDS

ALL *questions may be attempted,*
full marks may be obtained by substantially complete answers to ***TWO*** *questions*

There are ***three*** *questions in total*

The questions carry equal weight

Candidates may bring into the examination any lecture notes made during the course and any materials that were distributed during the course

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A layer of fresh ice occupying the region $a(t) < z < b(t)$ is sandwiched between a very deep layer of fresh water at temperature T_m occupying $z < a(t)$ and a very deep layer of salty water occupying $z > b(t)$, where z is measured vertically downwards. The salty water has far-field concentration C_0 and temperature T_∞. Its liquidus temperature is given by

$$T_L(C) = T_m - mC$$

where T_m and m are constants and $C(z,t)$ is the local salt concentration.

Analyze the evolution of the ice layer for the case $T_L(C_0) < T_\infty < T_m$, assuming that there is no convection in either of the liquid regions and that the temperature field in the ice layer is linear. In particular, find a similarity solution with $a = 2\lambda_a\sqrt{Dt}$, $b = 2\lambda_b\sqrt{Dt}$ and show that

$$\begin{aligned} 2\frac{L}{c_p}\epsilon^2\lambda_a &= -\frac{T_m - T_b}{\lambda_b - \lambda_a} \\ \frac{L}{c_p} &= \frac{T_b - T_\infty}{F(\epsilon\lambda_b)} - \frac{T_m - T_b}{2\epsilon^2\lambda_b(\lambda_b - \lambda_a)} \\ F(\lambda_b) &= \frac{C_b - C_0}{C_b} \end{aligned}$$

where

$$F(x) = \sqrt{\pi}\, x\, \mathrm{e}^{x^2}\, \mathrm{erfc}\, x,$$

$T_b = T_L(C_b)$ is the temperature at $z = b$, L is the specific latent heat of fusion of ice, c_p is the specific heat capacity and $\epsilon = \sqrt{D/\kappa}$, where D is the solutal diffusivity and κ is the thermal diffusivity. Note that λ_a and λ_b are both negative.

Show that, when $\epsilon \ll 1$, these equations are consistent with $\lambda_b = \mathrm{O}(1)$ and $\lambda_a = \mathrm{O}(\epsilon^{-1})$. In particular, show that in this limit

$$T_b - T_\infty \approx \frac{T_m - T_\infty}{1 - 2\epsilon\lambda_a/\sqrt{\pi}}$$

and

$$4\epsilon\lambda_a \approx \sqrt{\pi} - \sqrt{\pi + 8/S},$$

where $S = L/c_p(T_m - T_\infty)$.

Draw sketches of T, C and $T_L(C)$ as they vary with z and on the phase diagram. Explain physically how the system behaves and why.

2 In a cold, dry atmosphere, ice sublimates (turns directly into water vapour) at a temperature T_s less than its melting temperature T_m. In general, T_s depends on the relative humidity of the air but assume in this question that it is a given constant.

Consider a deep layer of ice of far-field temperature $T_\infty < T_s$ occupying $z > 0$, where z is measured vertically downwards, sublimating at its upper surface $z = 0$ as the sun shines on it. Model the solar radiation as providing an internal heat source per unit volume

$$q = q_0 \mathrm{e}^{-\lambda z}$$

in the ice, where q_0 and λ are both constant and z is measured in a frame of reference fixed with the sublimating upper surface ($z = 0$) of the ice.

Find the steady-state temperature distribution in the ice and determine that the sublimation rate is given by

$$V = \frac{q_0/\lambda}{\rho L + \rho c_p (T_s - T_\infty)},$$

where ρ is the density, L the specific latent heat of sublimation and c_p the specific heat capacity of the ice.

Sketch the temperature field and explain (without detailed calculation) the circumstances under which the ice will begin to melt and where melting will begin.

Analyse the linear morphological stability of the sublimating surface assuming that the heat source is unperturbed and approximating the steady temperature field by

$$T = T_s + G\,z,$$

where G is the steady-state temperature gradient at $z = 0$. Show, in particular, that the growth rate σ of perturbations with horizontal wavenumber α is given approximately by

$$\sigma = \frac{k}{\rho L}\alpha\left(G - \Gamma\alpha^2\right)$$

when the Stefan number $S = L/c_p(T_s - T_\infty)$ is large, where k is the thermal conductivity and Γ is the Gibbs-Thompson parameter. What is the condition on α for this approximate expression to be valid?

3 Hot magma flows turbulently up a long volcanic dike (a two-dimensional vertical channel in the Earth's crust), which initially has constant width b_0. The magma enters the dike with a constant volume flux Q_0 and temperature T_0. The crust has uniform temperature $T_m < T_0$, where T_m is the melting temperature of the crustal material.

Derive equations governing conservation of mass, and heat of the magma, and conservation of heat at the dike wall in terms of the width of the dike $b(z,t)$, the horizontal mean temperature of the magma $T(z,t)$ and the turbulent, mean vertical velocity $w(z,t)$. You can assume that the rate of turbulent heat transfer between the magma and the wall of the dike is given by

$$\alpha_T \rho c_p w(T - T_m),$$

where ρ is the density and c_p the specific heat capacity of the magma, and α_T is a constant heat transfer coefficient.

After a long time the dike width is much greater than b_0 and an approximate solution near the base of the dike can be found using a boundary condition that $b \to 0$ as $z \to \infty$. In this case, show that your equations and boundary conditions admit a similarity solution and determine the ordinary differential equations governing its behaviour.

Suggest in general terms (without giving specific algorithms) how you might solve these equations numerically.