UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 31 May 2002 9 to 11

PAPER 54

ELASTIC WAVES

*Attempt **THREE** questions*

*There are **five** questions in total*

The questions carry equal weight

[The stress-strain relation of linear elasticity is

$$\sigma_{ij} = c_{ijkl} e_{kl}$$

and the equation of motion is

$$\frac{\partial \sigma_{ij}}{\partial x_j} = \rho \frac{\partial^2 u_i}{\partial t^2} - \rho f_i \ ,$$

where ρ is density and f body force.]

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 The equations governing elastodynamic ray paths $\mathbf{x}(s)$ in an isotropic medium, where s is path length, are

$$\frac{d\mathbf{x}}{ds} = \alpha\mathbf{y} \ , \qquad \frac{d\mathbf{y}}{ds} = \operatorname{grad}(1/\alpha) \ ,$$

where $\alpha(\mathbf{x})$ is the wave speed. Give a physical interpretation of the vector $\mathbf{y}$.

Show that rays in material whose properties vary with depth only lie in a plane and satisfy Snell's law. Derive the corresponding result for a model Earth whose properties vary with radius only.

Find the relationship between the travel time T and the epicentral (angular) distance $\triangle$ of P-waves from a surface source to a surface receiver for a homogeneous spherical model Earth of radius R.

The model Earth is now changed to have a homogeneous core of radius R' surrounded by a homogeneous mantle with higher wave speed. Find the $T(\triangle)$ relation for the waves reflected from the core-mantle boundary and sketch the $T(\triangle)$ curves for both direct and reflected P.

Indicate with a diagram the paths of the rays refracted through the core and out again to a receiver on the surface.

2 Define what is meant by an elastic deformation for a material with Cauchy stress $\boldsymbol{\sigma}$ and Cauchy (infinitesimal) strain $\mathbf{e}$.

Assuming that energy is conserved in the deformation and that the strain energy is $\mathcal{E}(\mathbf{e})$ per unit mass, show that

$$\sigma_{ij} = \rho \frac{\partial \mathcal{E}}{\partial e_{ij}} \ ,$$

where ρ is the density, and that the stress-strain relation (for small strains) is approximately linear. Show also that there are 21 independent coefficients in general.

How many independent coefficients does a material have which has a 2-fold axis of symmetry; i.e. the material properties are unchanged if the material is rotated by π about the symmetry axis?

3 Show that there are, in general, three possible slownesses for a plane wave travelling in a given direction in a homogeneous anisotropic elastic material.

Define the slowness surface. Construct the energy flux vector and show that it is directed along the normal to the slowness surface.

Define the wave surface and explain how it may be used to predict the form of a disturbance from a point source.

Sketch typical cross-sections of both types of surface showing how cusps may arise on the wave surface.

4 A plane elastic pulse, with displacements $\mathbf{u}^0(\mathbf{x}, t)$ and wave front $t = \boldsymbol{\nu}\cdot \mathbf{x}/v$, where $\boldsymbol{\nu}$ is a unit vector and v is the wave speed, impinges on a cavity with smooth boundary S in an otherwise unbounded, homogeneous elastic medium. The resulting displacements can be written as $\mathbf{u} = \mathbf{u}^0 + \mathbf{u}^s$, where $\mathbf{u}^s(\mathbf{x}, t)$ is the scattered wave field. Write down the equations for $\mathbf{u}^s$.

Derive the uniqueness theorem for a structure of this kind and show, in particular, that the solution for $\mathbf{u}^s$ is unique.

5 Consider the problem of a seismic source in a heterogeneous and aspherical Earth model. Define the elastodynamic Green's function $\mathbf{G}(\mathbf{x}, \boldsymbol{\xi}, t)$ for the problem and show that, if the source consists of a body force distribution $\mathbf{f}(\mathbf{x}, t)$ within a region V, the resulting displacements can be written as

$$u_i(\mathbf{x}, t) = \int_0^t d\tau \left\{ \int_V f_j(\boldsymbol{\xi}, \tau) G_i^j(\mathbf{x}, \boldsymbol{\xi}, t-\tau) \rho \, dV_\xi \right\} .$$

Explain what is meant by a dipole source which gives rise to radiation of the form $\partial G_i^j(\mathbf{x}, \boldsymbol{\xi}, t)/\partial \xi_k$. Suppose that the seismic source consists of discontinuities in displacements $\mathbf{u}$ and stresses $\boldsymbol{\sigma}$ across a surface Σ. Show that the radiation is equivalent to a distribution of forces and dipoles over Σ.

What is the equivalent dipole source for a slip (discontinuity in transverse displacement) on a fault in isotropic material?