UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 6 June 2002 1.30 to 3.30

PAPER 52

ENVIRONMENTAL FLUID DYNAMICS

Attempt ***BOTH*** *questions from Section A,*
ONE *question from Section B*
and ***ONE*** *question from Section C*

There are ***two*** *questions in Section B and* ***two*** *questions in Section C*

Questions in Section A carry half the weight of questions in Section B and C

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

Section A

Answer both questions from Section A

1 Consider a suspension of small particles settling in a tall, narrow container of nominally quiescent fluid. Suppose that the settling velocity of the particles obeys $V = -V_0(1 - \phi/\phi_{max})^{\alpha}$ for some constant $\alpha \geq 0$, where V_0 is the settling velocity of an isolated particle, ϕ is the particle concentration and ϕ_{max} is the maximum possible concentration.

(a) Derive the evolution equation relating the vertical coordinate z, time t and particle concentration ϕ for this system, and demonstrate that the equation is hyperbolic.

(b) For each of the two specific initial conditions

$$\frac{\phi}{\phi_{max}} = \begin{cases} 0.8 & z > 1 \\ 0.5 + 0.3z & -1 \leq z \leq 1 \\ 0.2 & z < -1 \end{cases}$$

and

$$\frac{\phi}{\phi_{max}} = \begin{cases} 0.2 & z > 1 \\ 0.5 - 0.3z & -1 \leq z \leq 1 \\ 0.8 & z < -1 \end{cases}$$

sketch and describe the evolution of the particle concentration field for the case of $\alpha = 1$. How does this vary with the value of α?

2 Consider a cubic enclosure of height, width and depth H with a single high-level opening of height h as shown in Figure 1.

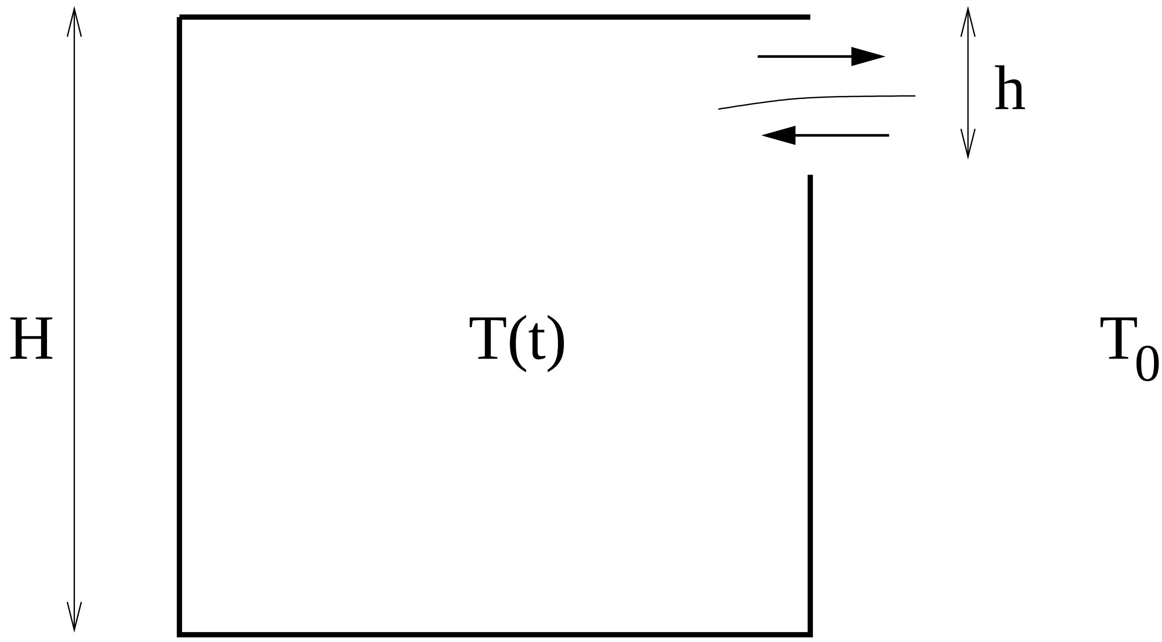

Figure 1: Cross-section of cubic enclosure of height, width and depth H.

At time $t = 0$ the temperature of the fluid in the enclosure is $T = T_1$ and the constant temperature of the exterior ambient fluid is $T = T_0$ with $T_1 > T_0$. There is an exchange flow across the opening with a warm upper layer of fluid of depth $h/2$ exiting the enclosure and a cool lower layer of depth $h/2$ entering the enclosure. The inlet and outlet speeds are given by $u_{in} = u_{out} = c(g'h)^{1/2}$ where c is a positive constant, $g' = g\beta\Delta T$ is the reduced gravity based on the temperature difference between the outlet and the inlet stream and β is the thermal expansion coefficient. Assume that turbulence maintains a uniform temperature in the enclosure and hence calculate $T(t)$ within the enclosure. What would you expect to happen if there was a single low-level opening?

Section B

Answer only one question from Section B.

3 Consider the high Reynolds number shallow water flow in a horizontal channel with a triangular cross-section described by the width

$$b(x, z) = B(x)z \quad \text{for} \quad z > 0,$$

where x is the downstream co-ordinate.

A layer of fluid of density ρ_f fills the channel to a depth $h(x, t)$ below an infinitely deep layer of density ρ_a. Gravitational acceleration g is oriented in the negative z direction.

(a) What restrictions must be placed on h, B and the fluid velocity $u(x, t)$ in order for the flow to satisfy the shallow water approximation? Derive the continuity equation for the layer.

(b) Describe the differences between the St. Venant and Benjamin front conditions, and where they may be applied. State any restrictions on the ratio ρ_a/ρ_f that may be appropriate. Define any parameters introduced. You need not derive the front conditions.

(c) State (without necessarily deriving) a suitable equation for momentum conservation, and determine the characteristics for the flow. Give a physical interpretation of these characteristics. You may ignore any drag terms. For the specific case $B(x) = 1$, determine the quantity conserved along the characteristics.

(d) Suppose that conditions are such that the St. Venant front condition is appropriate in a channel with $B(x) = 1$. If at $t = 0$ the channel is filled with a layer of density ρ_f to a depth $h = h_0$ for $x \leq 0$, and $h = 0$ for $x > 0$, determine the speed of the front and the rarefaction wave. Use a space-time plot to illustrate their progress, and sketch any characteristics that may help illustrate the flow. You need not derive expressions for $u(x, t)$ or $h(x, t)$.

(e) Consider also the release at one end of a semi-infinite $B(x) = 1$ channel of a finite volume V_0 of fluid for which the Benjamin front condition is appropriate. Derive and solve an integral model for the length $L(t)$ of the current.

4 Consider a steady, Boussinesq, particle-laden, shallow water hydraulic flow along a horizontal channel of uniform width. The bottom of the channel is porous and continuously draws in the fluid above it at a velocity W. The particles remain well mixed throughout the depth of the flowing layer, and there is no entrainment of ambient fluid into the flowing layer. Sedimentation of the particles onto the bottom does not affect the permeability or elevation of the bottom. At the start of the channel $x = 0$, the flowing layer is supplied with fluid containing a normalised particle concentration $\phi = 1$, corresponding to a reduced gravity g_0', at a flow rate Q_0 per unit width over a depth h_0. The particle settling velocity may be taken as a constant V, and the particle volume fraction is negligible.

(a) Derive the equations governing the depth h, particle concentration ϕ and velocity u along the channel. Calculate the along-channel variation in the volume flow rate Q and particle concentration. How far does the current extend?

(b) In the limit of small Froude number at $x = 0$, and the absence of particle settling ($V = 0$), determine the depth of the flow, and verify that the Froude number remains small.

(c) For settling particles when the bottom is not porous (and hence $W = 0$), calculate the depth profile in the limit of small Froude number at $x = 0$. Calculate also the profile in the limit of large Froude number at $x = 0$.

Section C

Answer only one question from Section C.

5 Consider a two-dimensional turbulent plume rising from a line source of heat of strength F. The flow obeys the two-dimensional turbulent thermal boundary layer equations

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0,$$

$$u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = g\beta(T - T_0),$$

$$u\frac{\partial T}{\partial x} + w\frac{\partial T}{\partial z} = 0,$$

where u, w are the mean horizontal and vertical velocities, x, z are the horizontal and vertical distances from the source, T is the mean temperature in the plume, $T_0 = T_0(z)$ is the temperature in the ambient, g is the gravitational acceleration and β is the thermal expansion coefficient.

(a) Derive an integral model using the standard entrainment hypothesis.

(b) Assume top-hat plume profiles and derive a system of ordinary differential equations which can be used to model two-dimensional turbulent plumes. State all assumptions.

(c) Consider the case of a uniform ambient of infinite horizontal extent with uniform temperature T_0. Solve the equations derived in (b) for the plume width $2b$, the plume velocity w and the plume reduced gravity $g' = g\beta(T - T_0)$ as a function of height z above the source.

(d) Suppose the plume is rising in the centre of a confined region which has horizontal width $2R$ and is initially filled with fluid of uniform temperature T_0. Describe the filling box mechanism and define the term "first front". Calculate the location of the first front as a function of time and determine the buoyancy jump across the first front. State all assumptions.

(e) Under what circumstances might you expect the filling box mechanism to break down?

6 A very tall vertical cylinder of radius R is initially filled with fluid of uniform density. There is a point source of buoyancy of strength B_o located at the centre of the closed base of the cylinder ($z = 0$) from which a pure turbulent plume rises. The plume has negligible volume flux at the source.

(a) Using the entrainment hypothesis and top-hat plume profiles, determine the equations governing the plume radius $b(z)$, vertical velocity $w(z)$ and reduced gravity $g'(z)$. [Hint: consider vertical flux balances, taking into account the motion of the ambient fluid.]

(b) Define the specific volume flux $Q(z)$, the specific momentum flux $M(z)$ and the specific buoyancy flux $B(z)$ for the plume in terms of b, w and g'. Deduce from the plume equations derived in (a) two ordinary differential equations for $Q(z)$ and $M(z)$. State any other conservation conditions that must be used.

(c) Show that $\frac{dM}{dz}$ has a singularity at some point $z = z_o$ above the source and at this point deduce the values of the plume radius, the vertical velocity in the plume, and the specific momentum flux in the plume and the environment. Give a physical interpretation of z_o and describe the flow you expect for $z > z_o$.