UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 30 May 2002 9 to 11

# PAPER 50

# MECHANICS OF COMPOSITES

*Attempt **THREE** questions*

*There are **five** questions in total*

*The questions carry equal weight*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1 (a)** Express the mean value $\overline{\varepsilon}_{ij}$ of the strain $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ over a domain $\Omega$ as an integral involving the displacement components $u_i$ over the boundary $\partial\Omega$. Express the mean value $\overline{\sigma}_{ij}$ of the self-equilibrated stress $\sigma_{ij}$ over $\Omega$ as an integral involving the traction components $t_i$ over the boundary $\partial\Omega$. Prove that

$$\frac{1}{|\Omega|}\int_{\Omega} \sigma_{ij}\varepsilon_{ij}\, d\boldsymbol{x} - \overline{\sigma}_{ij}\overline{\varepsilon}_{ij} = \frac{1}{|\Omega|}\int_{\partial\Omega} (\sigma_{ij} - \overline{\sigma}_{ij}) n_j (u_i - \overline{\varepsilon}_{ik} x_k)\, dS.$$

**(b)** The effective energy density function $W^{\mathrm{eff}}$ of an inhomogeneous body, with local energy density function $W(\varepsilon, \boldsymbol{x})$, occupying a domain $\Omega$, is defined to be

$$W^{\mathrm{eff}}(\overline{\varepsilon}) = \frac{1}{|\Omega|}\int_{\Omega} W(\varepsilon, \boldsymbol{x})\, d\boldsymbol{x},$$

where $\varepsilon_{ij}$ is the strain field induced in $\Omega$ when the boundary condition $u_i = \overline{\varepsilon}_{ik} x_k$ is applied over $\partial\Omega$. Prove that the mean stress and mean strain are related so that

$$\overline{\sigma}_{ij} = \frac{\partial W^{\mathrm{eff}}(\overline{\varepsilon})}{\partial \overline{\varepsilon}_{ij}}. \qquad (*)$$

**(c)** In the Voigt approximation, $\varepsilon_{ij}$ is taken to be uniform and so equal to $\overline{\varepsilon}_{ij}$. Prove that the mean stress $\overline{\sigma}_{ij}$, calculated directly from this prescription, conforms to $(*)$, with $W^{\mathrm{eff}}(\overline{\varepsilon})$ replaced by its Voigt approximation

$$W_V(\overline{\varepsilon}) = \overline{W}(\overline{\varepsilon}) = \frac{1}{|\Omega|}\int_{\Omega} W(\overline{\varepsilon}, \boldsymbol{x})\, d\boldsymbol{x}.$$

Prove the analogous result for the Reuss approximation

$$W_R(\overline{\varepsilon}) = (\overline{W^*})^*(\overline{\varepsilon}),$$

which is obtained by taking $\sigma_{ij} = \overline{\sigma}_{ij}$.

**2** Give the conditions of continuity that must apply at a perfectly bonded interface with normal $\boldsymbol{n}$ between two elastic media.

A laminate is composed of $N$ different elastic materials, material of type $r$ having elastic constant tensor $\boldsymbol{C}^r$ $(r = 1, 2 \cdots N)$. All interfaces have normal $\boldsymbol{n}$. The objective is to construct equilibrium stress and strain fields that take constant values $\sigma_{ij}^r$, $\varepsilon_{ij}^r$ in material $r$, and to deduce the effective elastic constant tensor $\boldsymbol{C}^{\mathrm{eff}}$ that relates their mean values $\overline{\sigma} = \sum_{r=1}^N c_r \sigma^r$ and $\overline{\varepsilon} = \sum_{r=1}^N c_r \varepsilon^r$, where $c_r$ is the volume fraction of material $r$. Introduce an additional, fictitious, material, with elastic constant tensor $\boldsymbol{C}^0$, and suppose that, if this material were also present in the laminate, the stress and strain within it would be $\sigma^0$, $\varepsilon^0$. Explain why it is possible to take

$$\varepsilon_{ij}^r = \varepsilon_{ij}^0 + (\alpha_i^r n_j + \alpha_j^r n_i)/2$$

for some $\alpha_i^r$. Deduce the relation

$$\varepsilon^r = [\boldsymbol{I} - \tilde{\boldsymbol{\Gamma}}^r(\boldsymbol{n})(\boldsymbol{C}^r - \boldsymbol{C}^0)]\varepsilon^0,$$

where

$$\tilde{\Gamma}_{ijkl}^r(\boldsymbol{n}) = n_i \left(K^r(\boldsymbol{n})\right)_{jk}^{-1} n_l \Big|_{(ij),(kl)}; \quad K_{ik}^r(\boldsymbol{n}) = C_{ijkl}^r n_j n_l.$$

Prove that

$$\tilde{\boldsymbol{\Gamma}}^r(\boldsymbol{C}^r - \boldsymbol{C}^s)\tilde{\boldsymbol{\Gamma}}^s = \tilde{\boldsymbol{\Gamma}}^s - \tilde{\boldsymbol{\Gamma}}^r$$

for any $r, s$.

Deduce that

$$\boldsymbol{C}^{\mathrm{eff}} = \left\langle \boldsymbol{C}[\boldsymbol{I} + \tilde{\boldsymbol{\Gamma}}^0(\boldsymbol{C} - \boldsymbol{C}^0)]^{-1} \right\rangle \left\langle [\boldsymbol{I} + \tilde{\boldsymbol{\Gamma}}^0(\boldsymbol{C} - \boldsymbol{C}^0)]^{-1} \right\rangle^{-1},$$

where $\langle f \rangle \equiv \overline{f} = \sum_{r=1}^N c_r f^r$. Show that this expression is independent of the choice of $\boldsymbol{C}^0$.

**3** Specialize the general representation

$$\boldsymbol{\Gamma}(\boldsymbol{x}) = \frac{-1}{8\pi^2}\int_{|\xi|=1} \tilde{\boldsymbol{\Gamma}}(\xi)\delta''(\xi.\boldsymbol{x})\,dS,$$

where $\tilde{\Gamma}_{ijkl}(\xi) = \xi_i(K^{-1}(\xi))_{jk}\xi_l|_{(ij),(kl)}$ with $K_{ik}(\xi) = C^0_{ijkl}\xi_j\xi_l$, to the case of an isotropic tensor $\boldsymbol{C}^0$, characterized by bulk modulus $\kappa^0$ and shear modulus $\mu^0$:

$$C^0_{ijkl} = \kappa^0\delta_{ij}\delta_{kl} + \mu^0(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - 2\delta_{ij}\delta_{kl}/3).$$

Calculate $\Gamma_{ijkk}(\boldsymbol{x})$ explicitly and deduce that

$$\Gamma_{iikk}(\boldsymbol{x}) = \frac{1}{\kappa^0 + 4\mu^0/3}\delta(\boldsymbol{x}),$$

where $\delta(\boldsymbol{x})$ is the three-dimensional Dirac delta.

An infinite isotropic matrix with bulk modulus $\kappa^0$ and shear modulus $\mu^0$ contains a bounded region $V$ within which the bulk modulus is $\kappa(\boldsymbol{x})$ but the shear modulus remains $\mu^0$. The matrix is loaded in such a way that, if $\kappa(\boldsymbol{x})$ were equal to $\kappa^0$ throughout $V$, the strain field would be $\varepsilon^0(\boldsymbol{x})$. Using the matrix as comparison material, show that the polarization within $V$ has the form $\tau_{ij}(\boldsymbol{x}) = \tau(\boldsymbol{x})\delta_{ij}$ for some scalar $\tau(\boldsymbol{x})$. Deduce that, within $V$,

$$\varepsilon_{kk}(\boldsymbol{x}) = \frac{3\kappa^0 + 4\mu^0}{3\kappa(\boldsymbol{x}) + 4\mu^0}\varepsilon^0_{kk}(\boldsymbol{x}),$$

and

$$\varepsilon_{ij}(\boldsymbol{x}) = \varepsilon^0_{ij}(\boldsymbol{x}) + \frac{1}{4\pi}\frac{\partial^2}{\partial x_i \partial x_j}\int_V \frac{3(\kappa(\boldsymbol{x}') - \kappa^0)}{3\kappa(\boldsymbol{x}') + 4\mu^0}\,\frac{\varepsilon^0_{kk}(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|}\,d\boldsymbol{x}'.$$

**4** The Hashin–Shtrikman variational approximation $\boldsymbol{C}^{HS}$ for the effective conductivity tensor of a polycrystal is given by

$$\boldsymbol{C}^{HS} = \left\langle [\boldsymbol{I} + (\boldsymbol{C} - \boldsymbol{C}^0)\boldsymbol{P}]^{-1} \right\rangle^{-1} \left\langle [\boldsymbol{I} + (\boldsymbol{C} - \boldsymbol{C}^0)\boldsymbol{P}]^{-1} \boldsymbol{C} \right\rangle ,$$

where $\langle \, \rangle$ denotes an average over all crystal orientations.

In two dimensions, the tensor $\boldsymbol{C}$ $(= C_{ij})$ has principal values $C_1$, and $C_2$, and $\boldsymbol{P} = \frac{1}{2C^0}\boldsymbol{I}$, with $\boldsymbol{C}^0 = C^0\boldsymbol{I}$, where $\boldsymbol{I}$ the two-dimensional identity tensor. Given that all orientations occur with equal frequency, prove that

$$\boldsymbol{C}^{HS} = \frac{2C_1C_2 + C^0(C_1 + C_2)}{C_1 + C_2 + 2C^0}\,\boldsymbol{I}.$$

Deduce the Hashin–Shtrikman lower and upper bounds ($C_l$, $C_u$ respectively) and show that the self-consistent approximation is

$$\boldsymbol{C}_{SC} = C_{SC}\boldsymbol{I}; \quad C_{SC} = (C_1C_2)^{1/2}.$$

Prove that $C_l \leqslant C_{SC} \leqslant C_u$.

**5** Prove the minimum energy principle in the form

$$\int_\Omega (W(\varepsilon) - \sigma_0 \varepsilon)\, d\boldsymbol{x} \leqslant \int_\Omega (W(\varepsilon') - \sigma_0 \varepsilon')\, d\boldsymbol{x},$$

where $\varepsilon$ is the actual strain field within an elastic solid occupying the domain $\Omega$, and $\varepsilon'$ is any candidate strain field, associated with a displacement that satisfies any displacement boundary conditions that may be specified on the boundary $\partial\Omega$. The energy density function $W$ is convex and $\sigma_0$ is any stress field that satisfies the equilibrium equations (with the given body force $\boldsymbol{f}$) and any prescribed traction boundary conditions on $\partial\Omega$.

The effective energy density function $W^{\text{eff}}(\overline{\varepsilon})$ of a composite with local energy function $W(\varepsilon, \boldsymbol{x})$, occupying a domain $\Omega$ (whose volume is normalized to 1) is defined as

$$W^{\text{eff}}(\overline{\varepsilon}) = \inf_{\varepsilon \in K} \int_\Omega W(\varepsilon, \boldsymbol{x})\, d\boldsymbol{x},$$

where $K$ denotes the set of strains compatible with the displacement boundary condition $u_i = \overline{\varepsilon}_{ik} x_k$ on $\partial\Omega$. Prove that, for any choice of $\tau(\boldsymbol{x})$ (with components $\tau_{ij}$) and any function $W_0(\varepsilon)$,

$$W^{\text{eff}}(\overline{\varepsilon}) \leqslant \inf_{\varepsilon \in K} \int_\Omega [\tau_{ij}\varepsilon_{ij} + W_0(\varepsilon) - (W - W_0)_*(\tau, \boldsymbol{x})]\, d\boldsymbol{x},$$

where

$$(W - W_0)_*(\tau, \boldsymbol{x}) = \inf_{\varepsilon} \{\tau_{ij}\varepsilon_{ij} - (W - W_0)(\varepsilon, \boldsymbol{x})\}.$$

Suppose that the composite has $n$ constituents, so that

$$W(\varepsilon, \boldsymbol{x}) = \sum_{r=1}^{n} W_r(\varepsilon)\chi_r(\boldsymbol{x}),$$

where $\chi_r(\boldsymbol{x})$ is the characteristic function of the region occupied by material $r$. Choosing $W_0(\varepsilon) = \frac{1}{2}\varepsilon \boldsymbol{C}^0 \varepsilon$ and

$$\tau(\boldsymbol{x}) = \sum_{r=1}^{n} \tau^r \chi_r(\boldsymbol{x}),$$

deduce the bound

$$W^{\text{eff}}(\overline{\varepsilon}) \leqslant \frac{1}{2}\overline{\varepsilon}\boldsymbol{C}^0\overline{\varepsilon} + \frac{1}{2}\sum_r c_r \tau^r_{ij}(\varepsilon_{ij} - \varepsilon^r_{ij}) + \sum_r c_r (W - W_0)_{**}(\varepsilon^r),$$

where $\varepsilon^r = (W_r - W_0)'_*(\tau^r)$ so that $(W_r - W_0)_*(\tau^r) + (W_r - W_0)_{**}(\varepsilon^r) = \tau^r_{ij}\varepsilon^r_{ij}$,

$$c_r \varepsilon^r + \sum_s \boldsymbol{A}_{rs}\tau^s = c_r \overline{\varepsilon}$$

and

$$c_r = \int_\Omega \chi_r(\boldsymbol{x})\, d\boldsymbol{x}, \quad \boldsymbol{A}_{rs} = \int_\Omega d\boldsymbol{x} \int_\Omega d\boldsymbol{x}' (\chi_r(\boldsymbol{x}) - c_r)\boldsymbol{\Gamma}(\boldsymbol{x}, \boldsymbol{x}')(\chi_s(\boldsymbol{x}') - c_s).$$

[*Recall that the operator* $\boldsymbol{\Gamma}$ *is such that* $\varepsilon(\boldsymbol{x}) = -\int_\Omega \boldsymbol{\Gamma}(\boldsymbol{x}, \boldsymbol{x}')\tau(\boldsymbol{x}')\, d\boldsymbol{x}'$ *is associated with a displacement field* $\boldsymbol{u}(\boldsymbol{x})$ *for which*

$$C^0_{ijkl} u_{k,lj} + \tau_{ij,j} = 0 \ \text{ in } \Omega, \quad u_i(\boldsymbol{x}) = 0 \ \text{ on } \partial\Omega.$$

*The properties* $\int_\Omega \boldsymbol{\Gamma}\, d\boldsymbol{x} = 0$, $\boldsymbol{\Gamma}\boldsymbol{C}^0\boldsymbol{\Gamma} = \boldsymbol{\Gamma}$, $\Gamma_{ijkl}(\boldsymbol{x}, \boldsymbol{x}') = \Gamma_{klij}(\boldsymbol{x}', \boldsymbol{x})$ *may be assumed.*]