UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 3 June 2002 9 to 12

PAPER 5

COMPLEX ANALYSIS

*Attempt **FOUR** questions*

*There are **six** questions in total*

The questions carry equal weight

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 State and prove the Schwarz - Pick Lemma.

The function $f : \mathbb{D}_R \to \mathbb{C}$ is analytic on some disc $\mathbb{D}_R = \{z \in \mathbb{C} : |z| < R\}$ with $R > 1$ and satisfies

$$A \leqslant |f(z)| \leqslant B \qquad \text{when} \quad |z| = 1 \ .$$

The constants A and B satisfy $|f(0)| < A \leqslant B < \infty$. Prove that f has a zero at some point $z_o \in \mathbb{D}_R$ and that $|z_o| \geqslant |f(0)|/B$. Show that there are functions f for which we obtain equality in this inequality.

2 What is a *normal family* of analytic functions? Show that the set of all analytic functions from a plane domain D into the unit disc $\mathbb{D}$ is a normal family.

Let $g : \mathbb{H}^+ \to \mathbb{C}$ be a bounded analytic function on the upper half-plane $\mathbb{H}^+$ and suppose that $g(z) \to \ell$ as z tends to ∞ along the positive imaginary axis. Show that the functions $z \mapsto g(tz)$ for $t \geqslant 1$ form a normal family. Deduce that, for each $\varepsilon > 0$, we have $g(z) \to \ell$ as z tends to ∞ in the sector

$$S(\varepsilon) = \{w \in \mathbb{H}^+ : \varepsilon < \arg w < \pi - \varepsilon\} \ .$$

Let $h : \mathbb{D} \to \mathbb{C}$ be a bounded analytic function and let ω be a complex number of modulus 1. Suppose that $h(r\omega) \to \ell$ as $r \nearrow 1$. Show that $h(z) \to \ell$ as z tends to ω in the region

$$\Sigma(k) = \{z \in \mathbb{D} : \text{ there exists } r \in [0,1) \text{ with } \rho(z, r\omega) < k\} \ .$$

Here ρ is the hyperbolic metric on the unit disc and k is an arbitrary positive constant.

3 State the Schwarz - Christoffel formula for conformal maps from the upper half-plane $\mathbb{H}^+$ onto a polygonal domain D. Explain the formula when D is the domain $S(\tau)$ obtained from $\mathbb{H}^+$ by cutting along the straight line segment from 0 to a point $\tau \in \mathbb{H}^+$.

Let $g : \mathbb{H} \to \mathbb{C}$ be the map $z \mapsto (z-1)^k(z+1)^{1-k}$ where $0 < k < 1$ and we choose the principal branches of the fractional powers. Show that g maps the upper half-plane conformally onto the domain $S(\tau)$ for some value of τ.

How is g related to the Schwarz - Christoffel map onto $S(\tau)$?

4 Explain how to define the hyperbolic Riemannian metric on any plane domain D that has the unit disc as its universal cover. Prove that it is well-defined and explain how it gives a metric on D. Calculate the hyperbolic metric on the annulus $A = \{z \in \mathbb{C} : 0 < |z| < 1\}$.

Prove Picard's Great Theorem. (You may assume the existence of a universal cover for the 3-punctured sphere.)

5 Describe how to construct an explicit universal cover of the 3-punctured sphere using the modular function.

6 Let A be a meromorphic function on the Riemann sphere taking values in the vector space of $N \times N$ complex matrices. Explains what it means for the differential equation

$$F' + A.F = 0 \qquad (*)$$

to have a regular singular point at the point z_o, which may be finite or ∞. Define the residue of A at a regular singular point. Show that, when all of the singular points of A are regular, then the sum of the residues is 0.

Suppose that A has a regular singular point at 0 with residue A_{-1}. Suppose also that 0 is the eigenvalue of A_{-1} with largest real part. Explain briefly why there is a non-constant analytic function F defined on a neighbourhood of 0 and satisfying the differential equation $(*)$.

By considering the example where

$$A(z) = \begin{pmatrix} 0 & 1 \\ 0 & -z^{-1} \end{pmatrix} ,$$

or otherwise, show that there need not always be a solution of the form $z^{\lambda} H(z)$ with H analytic on a neighbourhood of 0 when λ is an eigenvalue of A_{-1} that does not have the largest real part.