UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 3 June 2002 9 to 12

PAPER 49

FUNDAMENTALS OF ATMOSPHERE–OCEAN DYNAMICS

*Attempt **THREE** questions*

*There are **four** questions in total*

The questions carry equal weight

Clarity and explicitness of reasoning will attract more credit than perfection of computational detail

(x, y, z) denotes right-handed Cartesian coordinates and (u, v, w) the corresponding velocity components; t is time; the gravitational acceleration is $(0, 0, -g)$ where g is a positive constant.
The fluid is always incompressible. 'Ideal fluid' always means that buoyancy diffusion can be neglected where relevant, as well as viscosity. N denotes the buoyancy frequency of a stratified fluid.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Starting from the Boussinesq momentum, mass-continuity and buoyancy equations for two-dimensional motion of an ideal stratified, non-rotating fluid with constant N, linearize the equations about rest and derive the dispersion relation for an internal gravity wave having the two-dimensional plane-wave structure $\exp(ikx + imz - i\omega t)$.

Denote the complex amplitude of the disturbance buoyancy-acceleration field $\sigma(x, z, t)$ by $\hat{\sigma}$, so that $\sigma = \mathrm{Re}\{\hat{\sigma}\exp(ikx + imz - i\omega t)\}$ for the plane wave. Similarly, denote by $\hat{\mathbf{u}} = (\hat{u}, \hat{w})$, $\hat{\boldsymbol{\xi}} = (\hat{\xi}, \hat{\zeta})$, and $\hat{p}$ the complex amplitudes of the disturbance velocity, displacement and pressure fields $\mathbf{u} = (u, w)$, $\boldsymbol{\xi} = (\xi, \zeta)$, and p. Derive formulae for $\hat{u}$, $\hat{w}$, $\hat{\xi}$, $\hat{\zeta}$, and $\hat{p}$ in terms of $\hat{\sigma}$. Show in a sketch how the disturbance fields are distributed in space at a given instant, in the case of a wave whose phase velocity is directed upward and rightward.

Derive a formula for the group velocity $\mathbf{c}_\mathrm{g}$ in terms of N, k, and m. Show how the phase and group velocities can be represented geometrically by the sloping sides of a right-angled triangle, whose hypotenuse is horizontal with length proportional to $N/(k^2+m^2)^{1/2}$. Indicate the direction of $\mathbf{c}_\mathrm{g}$ in the sketch of disturbance fields.

Denote an average with respect to x by an overbar, and differentiation by suffixes. You may assume that

$$\left(\overline{\xi_x u} + \overline{\zeta_x w}\right)_t + \left(\overline{\zeta_x p}\right)_z = \nu\overline{\xi_x \nabla^2 u} + \nu\overline{\zeta_x \nabla^2 w} \qquad (*)$$

for waves subject to no buoyancy diffusion but to a viscous force $\nu(\nabla^2 u, \nabla^2 w)$ per unit mass, where ν is constant. On the assumption that ν is small enough for the plane-wave structure to remain a good approximation, show that the right-hand side is equal to $-\nu(k^2 + m^2)\left(\overline{\xi_x u} + \overline{\zeta_x w}\right)$. On the same assumption, show that $\overline{\zeta_x p} = \mathbf{c}_\mathrm{g} \cdot \hat{\mathbf{z}}\left(\overline{\xi_x u} + \overline{\zeta_x w}\right)$, where $\hat{\mathbf{z}}$ is the unit vertical vector. Deduce that if the wave field is steady, so that the time derivative in $(*)$ vanishes, then $|\hat{\sigma}|$ must be proportional to $\exp\left\{-\frac{1}{2}\nu(k^2 + m^2)z/(\mathbf{c}_\mathrm{g} \cdot \hat{\mathbf{z}})\right\}$. For the wave in your sketch, what conclusion can be drawn about the altitude z of the wave source, and how could the same conclusion have been drawn directly from the formula for $\mathbf{c}_\mathrm{g}$?

Comment *briefly* (a) on how the foregoing can be generalized to cases in which dissipating waves propagate on a mean flow $\bar{u}(z)$, and (b) on the implications for the altitude range affected by the waves when $\bar{u}(z)$ is such that Doppler shifting causes $|\mathbf{c}_\mathrm{g} \cdot \hat{\mathbf{z}}|$ to decrease toward zero.

2 Write down the shallow-water equations for the equatorial β-plane model, with Coriolis parameter $f = \beta y$, where $\beta =$ constant and $y = 0$ is the equator.

Linearize the equations about a basic state of rest relative to the earth with constant layer depth h_0. Make the equations dimensionless by rescaling the variables with respect to horizontal length scale $L = (c_0/\beta)^{1/2}$ for the x and y dependence, vertical scale h_0 for the surface elevation, and time scale L/c_0, where $c_0 = (gh_0)^{1/2}$ is the gravity-wave speed of the basic state.

Find the equatorially trapped waveguide modes, in which the disturbance velocity and surface-elevation fields take the form $\text{func}(y)\exp(ikx - i\omega t)$, with appropriate functions of y for the different fields. Show first that there is an eastward-travelling equatorially-trapped mode whose northward velocity component $v = 0$ everywhere, the other disturbance fields being proportional to $\exp\left(-\frac{1}{2}y^2\right)$ in dimensionless variables.

Show further that modes with $v \neq 0$ have $v = \hat{v}(y)\exp(ikx - i\omega t)$ where the real-valued function $\hat{v}(y)$ satisfies

$$\frac{d^2\hat{v}}{dy^2} + \left(\omega^2 - k^2 - \frac{k}{\omega} - y^2\right)\hat{v} = 0$$

in dimensionless variables. [*Hint: eliminate u first and surface elevation second.*] Deduce that all such equatorially trapped modes have the structure $\hat{v} \propto H_n(y)\exp(-\frac{1}{2}y^2)$ and satisfy the dispersion relation

$$\omega^2 - k^2 - \frac{k}{\omega} = 2n + 1 \qquad (n = 0, 1, 2, 3, ...) \ ,$$

where the Hermite polynomials $H_0 = 1,\ H_1 = 2y,\ H_2 = 4y^2 - 2,\ H_3 = 8y^3 - 12y$ etc. You may use the fact that the Hermite polynomials are defined such that $H_n'' - 2yH_n' + 2nH_n = 0$.

Show that the dispersion relation factorizes when $n = 0$, and that only the root for which

$$\omega - k - \frac{1}{\omega} = 0$$

corresponds to an equatorially trapped solution.

3 A stratified, rotating, Boussinesq fluid has constant buoyancy frequency N, and the frame of reference rotates with constant angular velocity $(0, 0, \frac{1}{2}f)$, $f > 0$. The fluid occupies the half-space $z > 0$. You may assume that quasi-geostrophic theory applies, such that the motion is governed by

$$\frac{D_g Q}{Dt} = 0 \qquad \text{with} \qquad Q = \left[f + \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{f^2}{N^2}\frac{\partial^2 \psi}{\partial z^2} \right]$$

in the standard notation of the lecture notes, with velocity field

$$(u,\, v,\, w) = \left\{ -\frac{\partial \psi}{\partial y}\,, \frac{\partial \psi}{\partial x}\,, -\frac{f}{N^2}\frac{D_g}{Dt}\left(\frac{\partial \psi}{\partial z}\right) \right\}$$

subject to the boundary condition $w = 0$ at $z = 0$. Find the linearized equation and boundary condition governing small disturbances $Q'(x, y, z, t)$, $\psi'(x, y, z, t)$ to a steady basic flow $\bar{u}(z) = \Lambda z$, $\bar{\psi}(z) = -\Lambda yz$, where Λ is a positive constant. Deduce that $Q' \propto F(y, z)G(x - \Lambda zt)$, where F and G are arbitrary functions.

When $F \equiv 0$ show that any disturbance streamfunction ψ' that depends on x and y according to $\psi' \propto \exp(ikx + ily)$, and is evanescent as $z \to \infty$, must have the vertical dependence

$$\psi' \propto \hat{\psi}(z) \equiv \exp(-\mu z) \quad \text{where} \quad \mu = \frac{N}{f}(k^2 + l^2)^{1/2}\,, \qquad (*)$$

regardless of the time-dependence of ψ'.

Show that propagating waves $\psi' \propto \hat{\psi}(z)\exp(ikx + ily - i\omega t)$ are possible solutions provided that

$$\omega = \frac{f\Lambda k}{N(k^2 + l^2)^{1/2}}\,. \qquad (\dagger)$$

Explain in physical terms, using sketches or otherwise, how the wave propagation mechanism works. Include some mention of how it depends on vortex-stretching on the height scale μ^{-1}.

A small but nonzero vertical velocity is imposed at the lower boundary, so that the boundary condition on the vertical disturbance velocity w' becomes

$$w' = \epsilon \cos(kx + ly - \omega_0 t) \qquad \text{at} \quad z = 0\,,$$

where ϵ is a small parameter. Using the result $(*)$, find solutions satisfying this boundary condition, (a) when $\omega_0 \neq \omega$, and (b) when $\omega_0 = \omega$, with ω given by $(\dagger)$.

Why does the solution grow without bound in case (b)?

4 Write an essay on the concept of potential-vorticity inversion, as illustrated by the four basic dynamical models (two-dimensional vortex dynamics, non-rotating layerwise-two-dimensional vortex dynamics, shallow-water quasi-geostrophic vortex dynamics, and stratified quasi-geostrophic vortex dynamics).