UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 5 June 2002 9 to 12

PAPER 47

SLOW VISCOUS FLOW

*Attempt up to **THREE** questions,*
*a distinction mark may be obtained by substantially complete answers to **TWO** questions*

*There are **four** questions in total*

The questions carry equal weight

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Use the Papkovich-Neuber representation of Stokes flow to derive the flow $\mathbf{u}_{\mathrm{G}}$ due to a couple $\mathbf{G}$ acting on a rigid sphere, radius a centred at $\mathbf{x} = \mathbf{0}$, in an unbounded fluid with no body forces.

State the Reciprocal Theorem for Stokes flows $\mathbf{u}_1$ and $\mathbf{u}_2$ with body forces $\mathbf{f}_1$ and $\mathbf{f}_2$, respectively.

(a) Apply the Reciprocal Theorem to the unbounded Stokes flow $\mathbf{u}(\mathbf{x})$ due to a body force $\mathbf{f}(\mathbf{x})$ acting on the fluid outside a rigid couple-free sphere and the flow $\mathbf{u}_{\mathrm{G}}$ derived above. Deduce that the couple-free sphere rotates with angular velocity

$$\boldsymbol{\Omega} = \frac{1}{8\pi\mu} \int_{r>a} \frac{\mathbf{x} \wedge \mathbf{f}(\mathbf{x})}{r^3}\, dV \ .$$

(b) Now consider the introduction of a rigid couple-free sphere centred at $\mathbf{x} = \mathbf{0}$ into an arbitrary unbounded Stokes flow $\mathbf{u}_\infty(\mathbf{x})$ with no body forces. Apply the Reciprocal Theorem to the perturbation flow and $\mathbf{u}_{\mathrm{G}}$ to deduce the Faxen formula

$$\boldsymbol{\Omega} = \frac{1}{2}\boldsymbol{\omega}_\infty(\mathbf{0}) \ , \tag{$*$}$$

where $\boldsymbol{\omega}_\infty(\mathbf{x}) = \boldsymbol{\nabla} \wedge \mathbf{u}_\infty$, for the rate of rotation $\boldsymbol{\Omega}$ of the sphere.

(c) Consider the introduction of a rigid force-free couple-free sphere at $\mathbf{x} = \mathbf{R}$ into the flow driven by another rigid sphere rotating with fixed angular velocity $\boldsymbol{\Omega}_0$ and with centre fixed at $\mathbf{x} = \mathbf{0}$; both spheres are of radius a and $a \ll R$. Use $(*)$ to calculate the leading-order approximation to the rate of rotation of the sphere at $\mathbf{R}$.

If the sphere at $\mathbf{R}$ is now acted on by a force $\mathbf{F}$, estimate the magnitude of the force required to keep the other sphere at $\mathbf{x} = \mathbf{0}$. Deduce that there is another leading-order contribution, of magnitude $O[(F/\mu a^2)(a/R)^3]$, to the rotation rate of the sphere at $\mathbf{R}$.

[*You may quote the result that the drag on a translating sphere is* $6\pi\mu a U$*.*]

2 The concentration C of insoluble surfactant on the surface of an inviscid bubble immersed in a very viscous fluid obeys the equation

$$\frac{DC}{Dt} = -C[\boldsymbol{\nabla}_s\boldsymbol{\cdot}\mathbf{u}_s + (\mathbf{u}\boldsymbol{\cdot}\mathbf{n})\boldsymbol{\nabla}_s\boldsymbol{\cdot}\mathbf{n}] + D_s\nabla_s^2 C\,, \qquad (\dagger)$$

where $\mathbf{n}$ is the unit normal out of the bubble; $\mathbf{u}_s = \mathbf{I}_s\boldsymbol{\cdot}\mathbf{u}$ and $\boldsymbol{\nabla}_s = \mathbf{I}_s\boldsymbol{\cdot}\boldsymbol{\nabla}$ are the tangential fluid velocity and tangential gradient operator respectively, where $(\mathbf{I}_s)_{ij} = \delta_{ij} - n_i n_j$ is the local projection tensor onto the interface. (Note $\mathbf{I}_s\boldsymbol{\cdot}\mathbf{n} = \mathbf{0}$.) Describe the physical interpretation of each of the terms in (†).

Consider the steady concentration $C(\mathbf{x})$ on a spherical bubble of radius a with an interfacial velocity $\mathbf{u} = \mathbf{I}_s(\mathbf{x})\boldsymbol{\cdot}\mathbf{F}\boldsymbol{\cdot}\mathbf{x}$, where $\mathbf{F}$ is a constant, symmetric, traceless second-rank tensor, and $\mathbf{x}$ is the position vector from the centre of the bubble. Under what condition on a, D_s and $|\mathbf{F}|$ is it possible to simplify (†) by writing $C = C_0 + C'$, where $|C'| \ll C_0$ and C_0 is uniform? Assuming that this condition is satisfied, show that

$$\boldsymbol{\nabla}_s\boldsymbol{\cdot}\mathbf{u}_s = -3\mathbf{n}\boldsymbol{\cdot}\mathbf{F}\boldsymbol{\cdot}\mathbf{n} \quad \text{and} \quad C' = A\mathbf{n}\boldsymbol{\cdot}\mathbf{F}\boldsymbol{\cdot}\mathbf{n}\ ,$$

where the constant of proportionality A should be found.

[*You may use the results* $\boldsymbol{\nabla}_s\mathbf{n} = \mathbf{I}_s/a$ *and* $\nabla_s^2(n_i n_j) = 2(\delta_{ij} - 3n_i n_j)/a^2$.]

For $C' \ll C_0$ the surface-tension coefficient is given by $\gamma(C) = \gamma_0 - \gamma_1 C'$, where $\gamma_0 = \gamma(C_0)$ and γ_1 is a positive constant. Viscous stresses and the variation of surface tension deform the shape of the drop slightly to $r = a\left(1 + \dfrac{\mathbf{x}\boldsymbol{\cdot}\mathbf{D}\boldsymbol{\cdot}\mathbf{x}}{r^2}\right)$, with curvature

$$\kappa = \frac{2}{a} + \frac{4}{a}\frac{\mathbf{x}\boldsymbol{\cdot}\mathbf{D}\boldsymbol{\cdot}\mathbf{x}}{a^2} + O(|\mathbf{D}|^2)\ ,$$

where $\mathbf{D}$ is a constant, symmetric, traceless second-rank tensor, and $|\mathbf{D}| \ll 1$. Write down the stress boundary condition for a fluid–fluid interface with surface tension γ and curvature κ, and show that in this case

$$[\boldsymbol{\sigma}\boldsymbol{\cdot}\mathbf{n}]_-^+ = \frac{2\gamma_0\mathbf{n}}{a} + \frac{4\gamma_0}{a}(\mathbf{n}\boldsymbol{\cdot}\mathbf{D}\boldsymbol{\cdot}\mathbf{n})\mathbf{n} + \frac{2A\gamma_1}{a}\big(\mathbf{I}_s\boldsymbol{\cdot}\mathbf{F}\boldsymbol{\cdot}\mathbf{n} - (\mathbf{n}\boldsymbol{\cdot}\mathbf{F}\boldsymbol{\cdot}\mathbf{n})\mathbf{n}\big)\ .$$

Assuming that $\mathbf{u} \to \mathbf{E}\boldsymbol{\cdot}\mathbf{x}$ as $r/a \to \infty$ and that $\mathbf{F} = \alpha\mathbf{E}$ and $\mathbf{D} = \beta\mathbf{E}$ in a steady state, explain why the Papkovich-Neuber potentials for the flow can be written in the form

$$\boldsymbol{\Phi} = \frac{Pa^3}{3}\mathbf{E}\boldsymbol{\cdot}\boldsymbol{\nabla}\frac{1}{r} \qquad \chi = \frac{1}{2}\mathbf{x}\boldsymbol{\cdot}\mathbf{E}\boldsymbol{\cdot}\mathbf{x} + \frac{Qa^5}{3}\mathbf{E}\,\boldsymbol{:}\boldsymbol{\nabla}\boldsymbol{\nabla}\frac{1}{r}\,,$$

where P and Q are constants. Given that these potentials correspond to

$$\mathbf{u} = (1 + P - 3Q)(\mathbf{n}\boldsymbol{\cdot}\mathbf{E}\boldsymbol{\cdot}\mathbf{n})\mathbf{x} + (1 + 2Q)\mathbf{I}_s\boldsymbol{\cdot}\mathbf{E}\boldsymbol{\cdot}\mathbf{x}$$

$$\boldsymbol{\sigma}\boldsymbol{\cdot}\mathbf{n} = 2\mu(1 - 3P + 12Q)(\mathbf{n}\boldsymbol{\cdot}\mathbf{E}\boldsymbol{\cdot}\mathbf{n})\mathbf{n} + 2\mu(1 + P - 8Q)\mathbf{I}_s\boldsymbol{\cdot}\mathbf{E}\boldsymbol{\cdot}\mathbf{n}$$

on $r = a$, show that in steady state the deformation of the bubble is given by

$$\mathbf{D} = \frac{5\mu\mathbf{E}a}{\gamma_0}\left(\frac{2 + M}{5 + 2M}\right),$$

where $M = A\gamma_1/\mu a$.

Show that $\alpha \to 0$ as $M \to \infty$ and interpret this result physically.

3 A long cylindrical tube of radius a and length L is immersed in a large volume of viscous fluid. The tube is held at a fixed position with its axis vertical, and is open at both ends so that it is both filled with and surrounded by fluid. A heavy close-fitting axisymmetric particle falls down the tube at velocity U with the symmetry axis of the particle coincident with the axis of the tube. The particle has length $2l$ and radius $a-h(z)$, $-l \le z \le l$, in cylindrical coordinates fixed in the particle, and $h \ll a, l \ll L$.

Explaining any approximations made, show that the flux Q out of the bottom of the tube is related to the pressure difference ΔP across the particle by

$$Q = \frac{\pi a^4 \Delta P}{8\mu L} \ . \tag{1}$$

Use lubrication theory to show that

$$\frac{\Delta P}{6\mu} = 2qI_3 - UI_2 \ , \tag{2}$$

$$\text{where} \ \ q = (\pi a^2 U - Q)/2\pi a \tag{3}$$

and $I_n = \int_{-l}^{l} h^{-n}\, dz$. By considering the forces acting on a suitable fluid control volume, show further that the upward force F on the particle is given by

$$F = \pi a^2 \Delta P + 2\pi a\mu(4UI_1 - 6qI_2). \tag{4}$$

Let dimensionless variables be defined by

$$Q^* = \frac{Q}{\pi a^2 U}, \quad \Delta P^* = \frac{\Delta P a}{6\mu U}, \quad q^* = \frac{q}{2aU}, \quad F^* = \frac{F}{6\pi\mu a U}, \quad I_n^* = a^{n-1} I_n, \quad \delta = \frac{3a}{4L}.$$

Express (1)–(4) in terms of these variables and solve for Q^*, ΔP^* and q^*. Hence obtain

$$F^* = \frac{I_3^* + \delta(\frac{4}{3}I_1^* I_3^* - I_2^{*2})}{1 + \delta I_3^*}$$

explaining any approximations made.

Consider the case of a spherical particle with radius $a(1-\epsilon)$ for which

$$I_1^* = \frac{\pi}{2}\frac{\sqrt{2\epsilon}}{\epsilon}, \qquad I_2^* = \frac{\pi}{4}\frac{\sqrt{2\epsilon}}{\epsilon^2} \quad \text{and} \quad I_3^* = \frac{3\pi}{16}\frac{\sqrt{2\epsilon}}{\epsilon^3}.$$

Deduce that F^* takes distinct asymptotic forms when $\delta \ll \epsilon^{5/2}$, $\epsilon^{5/2} \ll \delta \ll \epsilon^{1/2}$ and $\epsilon^{1/2} \ll \delta$, and calculate the leading-order approximations.

By considering the size of Q^*, q^*, ΔP^* and F^*, describe the dominant flow pattern and force balance in each of the three regimes.

4 Consider incompressible viscous flow in a two-dimensional rigid porous medium of uniform isotropic permeability k, occupying the (x, y) plane, in which a narrow straight crack of thickness $h(x)$ is embedded along $y = 0$ $(-a \le x \le a)$, where $a \gg h \gg k^{1/2}$. Far from the crack the flow is uniform and the Darcy velocity $\mathbf{u} = (U, 0)$. Fluid can enter and leave the crack through the porous walls, but the walls can be assumed to impose a no-slip boundary condition on the tangential component of velocity in the crack.

State Darcy's Law and show that the pore pressure p in the porous medium is harmonic. Derive the boundary condition

$$\frac{\partial}{\partial x}\left(\frac{h^3}{12}\frac{\partial p}{\partial x}\right) + k\left[\frac{\partial p}{\partial y}\right]_-^+ = 0 \quad \text{on} \quad y = 0 \quad (-a \le x \le a),$$

where $[\ ]_-^+$ denotes the jump across $y = 0$, briefly explaining any approximations made. [*Derivation of lubrication theory is* **not** *required.*] State the other boundary condition satisfied by p.

Ellipsoidal coordinates are defined by $x = a\cosh\xi\cos\eta$ and $y = a\sinh\xi\sin\eta$, where $0 \le \xi \le \infty$ and $0 \le \eta \le 2\pi$. Show on a rough sketch the curves $\xi = 0$, $\xi = 1$ and $\eta = n\pi/4$ $(n = 0, \ldots, 7)$. Derive the equations

$$\frac{\partial}{\partial \eta}\left(\frac{h^3}{\sin\eta}\frac{\partial p}{\partial \eta}\right) + 12ak\left(\left.\frac{\partial p}{\partial \xi}\right|_{\eta} + \left.\frac{\partial p}{\partial \xi}\right|_{2\pi-\eta}\right) = 0 \quad \text{on} \quad \xi = 0 \quad (0 \le \eta \le \pi),$$

$$\left(\frac{\partial p}{\partial \xi}, \frac{\partial p}{\partial \eta}\right) \sim \frac{\mu U a e^{\xi}}{2k}(-\cos\eta, \sin\eta) \quad \text{as} \quad \xi \to \infty.$$

[*You may assume that*

$$\frac{\partial}{\partial x} = \frac{a}{\Delta}\left(\sinh\xi\cos\eta\frac{\partial}{\partial \xi} - \cosh\xi\sin\eta\frac{\partial}{\partial \eta}\right), \quad \frac{\partial}{\partial y} = \frac{a}{\Delta}\left(\cosh\xi\sin\eta\frac{\partial}{\partial \xi} + \sinh\xi\cos\eta\frac{\partial}{\partial \eta}\right)$$

and $\nabla^2 = \dfrac{1}{\Delta}\left(\dfrac{\partial^2}{\partial \xi^2} + \dfrac{\partial^2}{\partial \eta^2}\right)$, *where* $\Delta = a^2(\sinh^2\xi + \sin^2\eta)$.]

For the case $h = H_0(1 - x^2/a^2)^{1/6}$, where H_0 is a constant, show that

$$p = (\mu U a/k)F(\xi; \alpha)\cos\eta\ ,$$

where $\alpha = H_0^3/24ka$ and F is to be found. Calculate the flux Q across the mid-point $x = 0$ of the crack, and find the limiting forms for $\alpha \gg 1$ and $\alpha \ll 1$. Give a physical interpretation of the parameter α and hence of these limiting forms.

Show that the far-field perturbation δp to the pressure caused by the presence of the crack is given in plane-polar coordinates by

$$\delta p \sim \frac{\alpha\mu U}{(\alpha+1)k}\frac{a^2\cos\theta}{r} \quad \text{as} \quad r \to \infty.$$

[*Hint: Evaluate the perturbation in ellipsoidal coordinates first.*] Hence calculate the change in dissipation due to the crack, as given by the line integral $-\int_C \delta p\, U n_x\, ds$, where C is a suitably chosen curve with outward normal $\mathbf{n} = (n_x, n_y)$.

A porous medium containing a random distribution of cracks, each identical in size, shape and orientation to the one analysed above, behaves like an anisotropic porous medium with effective permeability tensor $\mathbf{k}^*$. Assume that the number of cracks per unit area ϕ is sufficiently small that they do not interact with each other. Calculate k^*_{xx} by comparing the dissipation per unit area $\mu U^2/k^*_{xx}$ for flow in a uniform medium with the average dissipation per unit area in the medium with cracks.