UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 6 June 2002 1.30 to 3.30

PAPER 46

ACCRETION DISCS

Attempt ***TWO*** *questions*

There are ***three*** *questions in total*

The questions carry equal weight

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (i) In a spherically symmetric gravitational potential, circular orbits are possible in any plane passing through the origin. Show that infinitesimal changes in the specific energy and specific angular momentum vector of a test particle, within the family of circular orbits, are related by

$$\mathrm{d}\varepsilon = \mathbf{\Omega}\cdot\mathrm{d}\mathbf{h},$$

where $\mathbf{\Omega}$ is the angular velocity vector of the orbit.

Two test particles are constrained to move on circular orbits in a spherically symmetric potential. They exchange infinitesimal quantities of mass and angular momentum, while conserving the total mass and angular momentum of the system. Show that the change in the total energy of the system is

$$\mathrm{d}E = [(\varepsilon_1 - \Omega_1 h_1) - (\varepsilon_2 - \Omega_2 h_2)]\,\mathrm{d}M_1 + (\mathbf{\Omega}_1 - \mathbf{\Omega}_2)\cdot\mathrm{d}\mathbf{H}_1,$$

where $\mathrm{d}M_1$ and $\mathrm{d}\mathbf{H}_1$ are the infinitesimal changes in the mass and angular momentum vector of particle 1. Hence argue that the exchanges most efficient in releasing energy are associated with an inward transfer of mass, an outward transfer of angular momentum and a reduction in the relative inclination of the two orbits. You may assume that $\Omega = |\mathbf{\Omega}|$ is a monotonically decreasing function of radius.

(ii) In a steady Keplerian accretion disc with inner radius r_{in} and accretion rate $\dot{M}$ the relation

$$\bar{\nu}\Sigma = \frac{\dot{M}}{3\pi}\left[1 - \left(\frac{r_{\text{in}}}{r}\right)^{1/2}\right]$$

holds. Compare the rate at which energy is dissipated by viscosity in an annulus $a < r < b$ with the rate at which orbital binding energy is acquired by gas in passing through the annulus. Show that these rates are in balance for the disc as a whole, but that the rate at which energy is dissipated in an annulus far from the inner edge ($a \gg r_{\text{in}}$) is approximately three times the rate at which binding energy is acquired by the gas in passing through it.

2 The vertical structure of a thin, Keplerian accretion disc with constant (Thomson) opacity and a mixture of gas and radiation is governed by the equations

$$\begin{aligned}
\frac{\partial p}{\partial z} &= -\rho\Omega^2 z, \\
\frac{\partial F}{\partial z} &= \mu\left(r\frac{d\Omega}{dr}\right)^2, \\
F &= -\frac{16\sigma T^3}{3\kappa\rho}\frac{\partial T}{\partial z}, \\
p &= \frac{k\rho T}{\mu_{\mathrm{m}} m_{\mathrm{H}}} + \frac{4\sigma T^4}{3c}.
\end{aligned}$$

(i) Explain briefly the physical meaning of each equation.

(ii) Show that there is only one possible value of the effective viscosity μ of an accretion disc in which the gas pressure is negligible compared to the radiation pressure.

(iii) According to one theory, whatever the ratio of gas and radiation pressures, the effective viscosity is always related to the gas pressure p_{g} by $\mu = \alpha p_{\mathrm{g}}/\Omega$, where α is a constant. Show that the temperature then satisfies an equation of the form

$$\frac{1}{\rho}\frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial T^4}{\partial z}\right) + A\Omega T = 0,$$

where A is a constant to be determined. By means of a suitable change of variables, find an explicit expression for the mean kinematic viscosity $\bar{\nu}(r, \Sigma)$ of the disc. You may assume that the 'zero boundary conditions' apply, and that there exists a unique non-trivial solution $t(\zeta)$ of the boundary-value problem

$$\frac{d^2 t^4}{d\zeta^2} + t = 0, \qquad t'(0) = t(1) = 0.$$

Your answer may involve an integral of $t(\zeta)$, which need not be evaluated.

3 Describe the construction of the shearing sheet as a local model of accretion disc dynamics. Assuming that the fluid is incompressible and of uniform density, viscosity and electrical conductivity, explain why the equation of motion and the induction equation may be written in the form

$$\left(\frac{\partial}{\partial t} - 2Ax\frac{\partial}{\partial y}\right)\mathbf{v} - 2Av_x\mathbf{e}_y + 2\boldsymbol{\Omega}\times\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} = -\nabla\psi + \frac{1}{\mu_0\rho}\mathbf{B}\cdot\nabla\mathbf{B} + \nu\nabla^2\mathbf{v},$$

$$\left(\frac{\partial}{\partial t} - 2Ax\frac{\partial}{\partial y}\right)\mathbf{B} + \mathbf{v}\cdot\nabla\mathbf{B} = -2AB_x\mathbf{e}_y + \mathbf{B}\cdot\nabla\mathbf{v} + \eta\nabla^2\mathbf{B},$$

where $\mathbf{v}$ is the velocity perturbation, and ψ is a certain function. State two further conditions satisfied by $\mathbf{v}$ and $\mathbf{B}$.

Show that there exist plane-wave solutions of the form

$$\mathbf{v} = \mathrm{Re}\left\{\tilde{\mathbf{v}}(t)\exp\left[\mathrm{i}\mathbf{k}(t)\cdot\mathbf{x}\right]\right\},$$
$$\psi = \mathrm{Re}\left\{\tilde{\psi}(t)\exp\left[\mathrm{i}\mathbf{k}(t)\cdot\mathbf{x}\right]\right\},$$
$$\mathbf{B} = \mathrm{Re}\left\{\tilde{\mathbf{B}}(t)\exp\left[\mathrm{i}\mathbf{k}(t)\cdot\mathbf{x}\right]\right\},$$

and obtain equations governing the evolution of $\tilde{\mathbf{v}}$, $\tilde{\mathbf{B}}$ and $\mathbf{k}$. By considering the rates of change of $|\tilde{\mathbf{v}}|^2$ and $|\tilde{\mathbf{B}}|^2$, or otherwise, show that all non-axisymmetric waves eventually decay.